Elementary Functions in TBA

JUNJI SUZUKI (SHIZUOKA UNIVERSITY)

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Aim how to interpret and generalize the result obtained by P. Fendley (hep-th 9906114) in view of ODE/IM

Thermodynamic Bethe Ansatz Equation

Gaudin, Takahashi, M. Suzuki, Al Zamolodchikov

Powerful machinery in $\begin{cases} \text{ID quantum system at finite T} \\ \text{1+1 D QFT at finite L} \end{cases}$

Input

S matrix

physical basis (strings)

Fusion relations

$\Rightarrow$

Output

Coupled nonlinear integral equations

Numerical Study possible

Nonlinearity $\rightsquigarrow$ Analytic Solution difficult

N=2 SUSY in 2D : TBA [Fendley-Intriligator '92]

$$2\mathcal{D}(\theta) = \mathcal{E}(\theta) + \int_{-\infty}^{\infty} \frac{\ln(1+\eta^2(\theta'))}{\cosh(\theta-\theta')}\frac{d\theta'}{2\pi} \qquad 2\mathcal{D}(\theta) = z\cosh\theta$$

$$\eta(\theta) = \int_{-\infty}^{\infty} \frac{e^{-\mathcal{E}(\theta')}}{\cosh(\theta-\theta')}\frac{d\theta'}{2\pi}$$

massless limit

Fact [Fendley'99] $\exists$ explicit solution $\qquad 2\mathcal{D}(\theta) = e^{\theta}$

$$e^{-\mathcal{E}(\theta)} = -2\pi \frac{d}{dz}\left(\mathrm{Ai}(z)\right)^2$$

$$\eta(\theta) = -2\pi \frac{d}{dz}\left[\mathrm{Ai}(z e^{\frac{\pi}{3}i})\,\mathrm{Ai}(z e^{-\frac{\pi}{3}i})\right] \qquad z = \left(\frac{3}{4}e^{\theta}\right)^{\frac{2}{3}}$$

Fendley's Proof

Use results by $\left\{\begin{array}{l}\text{McCoy Tracy Wu} \\ \text{Al Zamolodochikov} \\ \text{Tracy-Widom}\end{array}\right.$ on P_{III}

theorem [TW]

$$\boxed{\begin{array}{l}\text{for integral operator } K(\theta,\theta') = 2\dfrac{E(\theta)E(\theta')}{e^{\theta}+e^{\theta'}} \quad E(\theta) := e^{\theta/2 - D(\theta)} \\ \text{Define } Q_+ = e^{-\frac{2\theta}{3}}(I+K)^{-1}E(\theta) \quad Q_- = e^{-\theta/3}(I-K)^{-1}E \\ \text{then } e^{-\varepsilon(\theta)} = 4\pi\, Q_+(\theta)\, Q_-(\theta)\end{array}}$$

Fendley's solution $\Rightarrow Q_+ = Ai(z) \quad Q_- = Ai'(z)$

So check via $\boxed{(I+K)e^{2\theta/3}Q_+ = E(\theta)}$

Any Simpler Way ?

Use ODE/IM

Plan of the talk

1. brief review on ODE/IM

2. Fendley-Intriligator TBA

3. ODE/IM interpretation

4. Generalization

5. Outlook

Brief review on ODE/IM

$$\left(-\frac{d^2}{dz^2}+V(z,E)\right)\psi(z,E)=0$$

Consider generally $z\in\mathbb{R}\longrightarrow z\in\mathbb{C}$

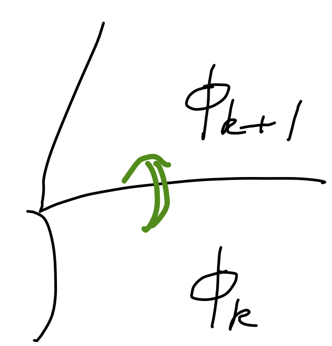
When Z= ∞ = irregular singularity

$\Rightarrow$ $\mathbb{C}$ plane is divided into sectors $S_k \ni (\phi_k, \psi_k)$

Fundamental set of solutions

Connection problem

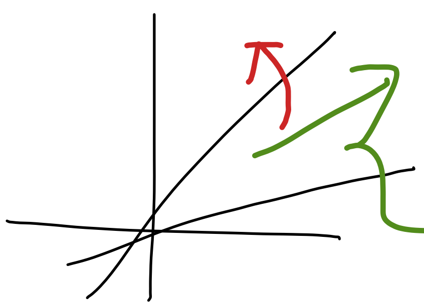
relate FSS of neighbouring sectors



Stokes phenomenon

Sudden discontinuous change in ψ as one crosses boarder

quantify $\rightarrow$ Stokes multiplier



$$\left\{ \begin{array}{ll} \psi_r & \text{recessive (subdominant)} \\ \psi_d & \text{dominant} \end{array} \right.$$

$$\left\{ \begin{array}{l} \psi_d \rightarrow \psi_d + \tau \psi_r \\ \psi_r \rightarrow \psi_r \end{array} \right.$$

The simplest case $(V(x) = x)$ τ is simply 1

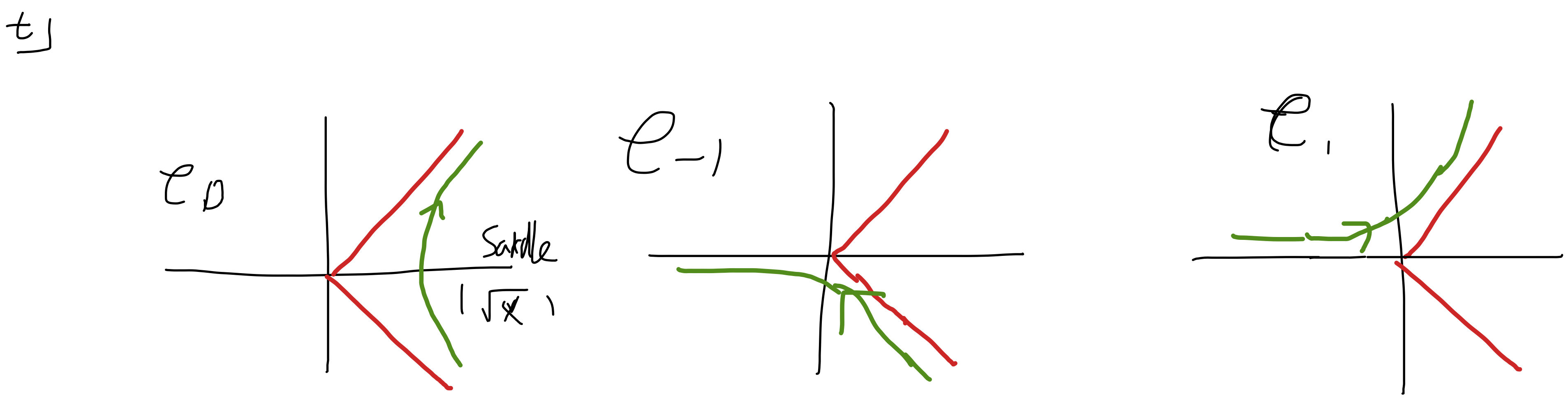
other cases τ is nontrivial

The simplest case $V(x) = x$.

explicit solution is known in this case (= Airy function)

$$\psi(x) = \int_{\mathcal{C}} dt\, e^{+\left(t^3/3 - tx\right)}$$ for arbitrary $\mathcal{C}$

Choices of Contours



ψ_i for $\mathcal{C}_i$

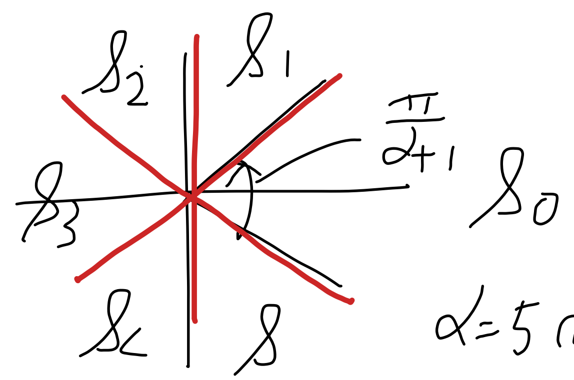
then $\psi_0 = \psi_{-1} + \psi_1 \implies \tau = 1$!!

Dorey-Tateo-Voros potential

$$V(z,E) = z^{2\alpha} - E$$

explicit ψ is not available but $\exists$ good property of ODE

S_k



$\alpha = 5$ case

$k \in \mathbb{Z}_{\alpha+1}$

Definition

$$\varphi(x) = \text{A solution} \sim x^{-\frac{\alpha}{2}} \exp\left(-\frac{x^{\alpha+1}}{\alpha+1}\right) \quad \text{in } S_0$$

Lemma I φ_k solves Schroedinger equation

$$\varphi_k(x,E) \equiv \omega^{k/2}\,\varphi(\omega^{-k}x,\ \Omega^{-k}E)\sqrt{\frac{1}{2i}}$$

$$\omega \equiv \exp\left(\frac{\pi i}{\alpha+1}\right) \qquad \Omega = \exp\left(\frac{2\alpha}{\alpha+1}\pi i\right)$$

Connection problem

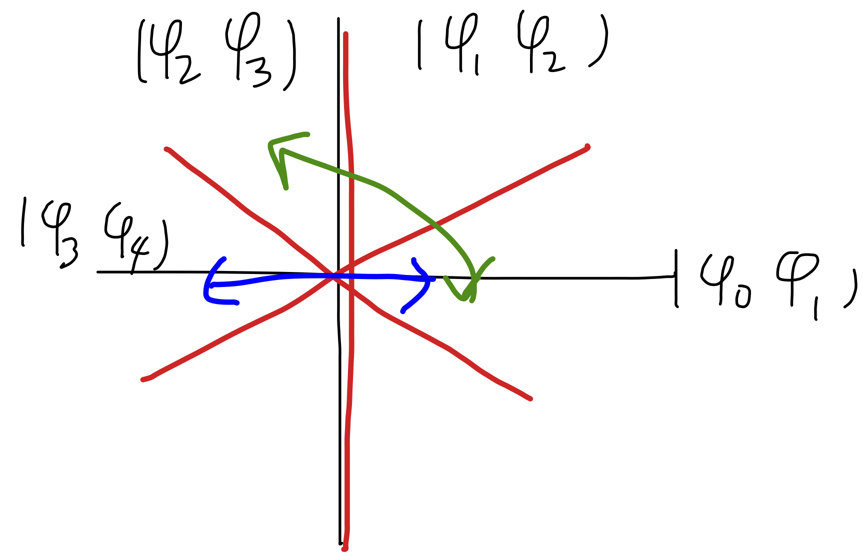
FSS in $\mathcal{S}_k$ $\begin{cases} \varphi_k & \text{sub-dominant} & \varphi_r \\ \varphi_{k+1} & \text{dominant} & \varphi_d \end{cases}$

Elementary Connection problem $(\mathcal{S}_0, \mathcal{S}_1)$

$$T(E)\varphi_1 = \varphi_0 + \varphi_2 \qquad \text{or} \qquad (\varphi_0 \ \ \varphi_1) = (\varphi_1 \ \ \varphi_2)\begin{pmatrix} T(E) & 1 \\ -1 & 0 \end{pmatrix}$$

generalized Connection problem

$$\begin{pmatrix} \varphi_0 & \varphi_1 \\ \varphi_0' & \varphi_1' \end{pmatrix} = \begin{pmatrix} \varphi_k & \varphi_{k+1} \\ \varphi_k' & \varphi_{k+1}' \end{pmatrix} M_k(E)$$



Generalized connection = T/Y -system

proposition $\quad M_k(E)$ is represented in the form

$$M_k(E) = \begin{pmatrix} \tau_k(E) & \tau_{k-1}(E\Omega^{-1}) \\ -\tau_{k-1}(E) & -\tau_{k-2}(E\Omega^{-1}) \end{pmatrix} \quad \text{with } \tau_0 = 1, \ \tau_{-1} = 0$$

τ_k has a Wronskian representation $\quad \tau_k(E) = \dfrac{W[\varphi_0 \ \varphi_{k+1}]}{W[\varphi_k \ \varphi_{k+1}]}$

$$T_k(E\Omega^{1/2})\, T_k(E\Omega^{-1/2}) = 1 + T_{k-1}(E)\, T_{k+1}(E)$$

Kluemper-Pearce transformation $\quad Y_k(E) := T_{k-1}(E)\, T_{k+1}(E)$

$$Y_k(E\Omega^{1/2})\, Y_k(E\Omega^{-1/2}) = (1 + Y_{k+1}(E))(1 + Y_{k-1}(E))$$

ODE/IM correspondence

Elementary connection problem $\simeq$ Baxter's TQ relation

$T \sim$ Stokes multiplier

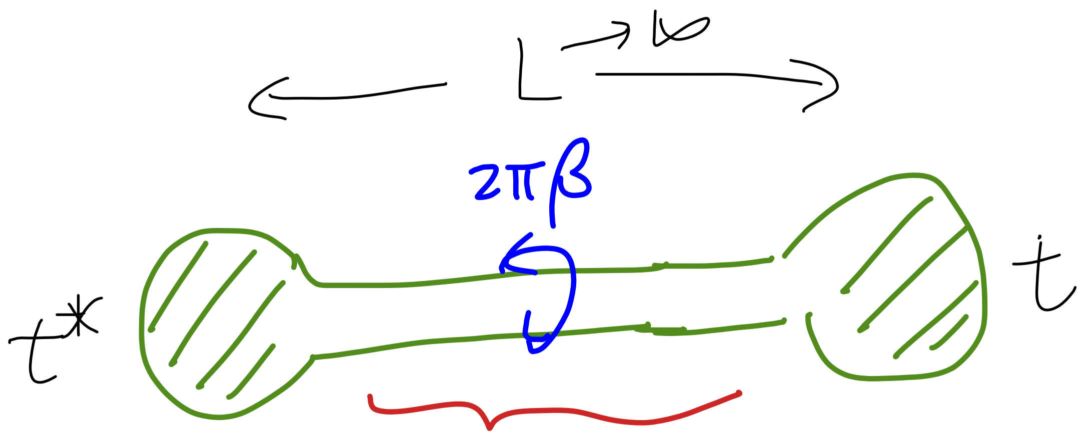
$Q \sim$ wave function

(generalized) Stokes multipliers $\rightarrow$ solution to Y-system

2. Fendley - Intriligator TBA

Problem gluing topological and anti-topological theories

Cecotti-Vafa



Superfield

$$\mathcal{L}_{int} = \int H(X_i\ \bar{X}_i)\, d^4\theta + \int W(X_i)\, d^2\theta + hc$$

D term F term

Choose $W(X_i) = \dfrac{X_i^3}{3} - tX_i$ $\Rightarrow$ (the simplest case of)

Fendley- Intriligator TBA

CFT limit may be c=0 theory

c=0 case of ODE/IM

$$V(x,E) = x^{2\alpha} - E$$

$$\Rightarrow \quad c = 1 - \frac{6\alpha^2}{\alpha+1} = 0 \quad \Rightarrow \quad \alpha = \frac{1}{2}$$

Dorey-Tateo solution

$$T_1(E) = \Omega^{\frac{1}{2}} \frac{Q(E\Omega^{-2})}{Q(E)} + \Omega^{-\frac{1}{2}} \frac{Q(E\Omega^{2})}{Q(E)} \qquad \Omega = e^{2\pi i/3}$$

$$Q(E) = Ai(E)$$

Connection rule of Airy function $\quad Ai(E) = \Omega\, Ai(E\Omega^{-2}) + \Omega^{-1} Ai(E\Omega^{2})$

$$\Rightarrow \quad T_1 = 1. \quad T_2 = 0 \; \ldots$$

Only trivial Y naively.

h-deformation

Idea (essentially due to Fendley)

Deform dressed Vacuum Form

$$T_1(E) = \xi \frac{Q(E\Omega^{-2})}{Q(E)} + \xi^{-1} \frac{Q(E\Omega^{2})}{Q(E)}$$

$$\xi = e^{\left(\frac{\pi - h}{3}\right)i} \qquad h \to 0$$ deformation by h

or $E = e^{2/3\,\theta}$

$$T_1(\theta) = \xi \frac{Q(\theta - \frac{\pi}{2} i)}{Q(\theta)} + \xi^{-1} \frac{Q(\theta + \frac{\pi}{2} i)}{Q(\theta)} \quad \text{---} \quad (\star)$$

and seek for the connection to Fendley's solution

Advantage: $h=0$ we know explicit T and Q

Drawback: $h \neq 0$ we do not Know T and Q

h deformation and Fendley's Solution

We will argue

- TBA can be analyzed order by order in h
- N= 2 SUSY TBA =first nontrivial eq in h
- Solution is fixed only by $O(h^0)$ information

We know everything

closed TBA

before h-deformation

$$T_1^{Airy} T_1^{Airy} = 1$$

$$T_2^{Airy} = 0$$

after h-deformation

$$\Rightarrow \quad T_1\left(\theta+\frac{\pi i}{2}\right) T_1\left(\theta-\frac{\pi i}{2}\right) = 1 + T_2(\theta)$$

$$T_2\left(\theta+\frac{\pi}{2}i\right) T_2\left(\theta-\frac{\pi}{2}i\right) = 1 + T_1(\theta) T_3(\theta)$$

$$\vdots$$

$$\infty$$

however

$$T_3(\theta) = \xi^3 + \xi^{-3} + T_1(\theta)$$

$\Rightarrow$ Functional relations close among T_1 and T_2 (BLZ)

$+$ Analyticity Assumption $=$ closed TBA eq

$$\ln T_1(\theta) = \int \frac{d\theta'}{2\pi} \ln\left(1 + T_2(\theta')\right) \frac{1}{\mathrm{ch}(\theta-\theta')} \quad \text{— ⓐ}$$

$$\ln T_2(\theta) = D(\theta) + \int \frac{d\theta'}{2\pi} \ln\left(1+\xi^3 T_1(\theta')\right)\left(1+\xi^{-3} T_1(\theta')\right) \frac{1}{\mathrm{ch}(\theta-\theta')} \quad \text{— } b$$

N =2 SUSY TBA as O(h) equation

As $T_1^{h=0}(\theta) = T_1^{\mathrm{Airy}} = 1 \qquad T_2^{h=0}(\theta) = T_2^{\mathrm{Airy}} = 0$

So reasonable to assume

$$T_1 = e^{-\varepsilon_t} = e^{h\eta(\theta)} = 1 + h\eta(\theta) + O(h^2)$$

$$T_2 = e^{-\varepsilon_1} = h e^{-\varepsilon(\theta)} + O(h^2)$$

(a) $\xrightarrow{O(h)}$ $$\eta(\theta) = \int \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta-\theta')} e^{-\varepsilon(\theta)}$$

(b) $$\cancel{\ln h} - \varepsilon(\theta) = D(\theta) + \int \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta-\theta')} \underbrace{\ln\left(1 - e^{h(i+\eta(\theta))}\right)\left(1 - e^{h(\eta(\theta)-i)}\right)}_{= \, \ln \cancel{h^2}(1+\eta^2)}$$

$$\rightarrow \varepsilon(\theta) = -D(\theta) - \int \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta-\theta')} \ln\left(1+\eta^2(\theta')\right)$$

$\mathcal{N} = 2$ TBA recovered !

(quantum) Wronskian and Solution to T-system

BLZ. quantum Wronskian

$$2i\sin 2\pi p\, T_j(\lambda) = e^{2\pi i(j+1)p} A_+(\lambda\Omega^{\frac{j+1}{2}}, p) A_-(\lambda\Omega^{-\frac{j+1}{2}}, p) - e^{-2\pi i(j+1)p} A_+(\lambda\Omega^{-\frac{j+1}{2}}, p) A_-(\lambda\Omega^{\frac{j+1}{2}}, p)$$

$j = 0, 1, 2.$

$P = \frac{1}{6} - \frac{h}{6\pi}$

$$A_\pm(\lambda) = \lambda^{\mp\frac{2P}{\beta^2}} Q_\pm(\lambda)$$

$=$ function of $\lambda^2 \ (\propto E)$

Dorey Tateo solution $(h=0)$

$$A_+(\lambda, \tfrac{1}{6}) = C_+ Ai(\lambda^2), \quad A_-(\lambda, \tfrac{1}{6}) = C_- \frac{d}{dz} Ai(z)\Big|_{z=\lambda^2}$$

Observation

By expanding both sides of quantum Wronskian, one easily obtains

Fendley's solution

$\frac{\partial A_\pm}{\partial p}$ do not contribute example: j=2 of q-Wronskian relation

$$\mathrm{LHS} = 2i\sin 2\pi p T_2 \simeq 2i\left(\sin\frac{\pi}{3} - \frac{h}{3}\cos\frac{\pi}{3}\right) h\, e^{-\varepsilon(\theta)}$$

$$= 2i\sin\frac{\pi}{3} e^{-\varepsilon(\theta)} h + O(h^2)$$

$$\mathrm{RHS} = e^{h_i}\left(A_+(\lambda q^{-3/2}) A_-(\lambda q^{3/2})\right)\Big|_{p=\frac{1}{6}} - e^{-h_i}\left(A_+(\lambda q^{3/2}) A_-(\lambda q^{-3/2})\right)\Big|_{p=\frac{1}{6}}$$

$$\underbrace{- \frac{h}{6\pi}\left[\frac{d}{dp}A_+(\lambda q^{-3/2}) A_-(\lambda q^{3/2}) - \frac{d}{dp}A_+(\lambda q^{3/2}) A_-(\lambda q^{-3/2})\right]}_{0\,!}$$

$$A_+(\lambda) = A_+(\lambda^2)$$
$$q^3 = 1$$

$$\mathrm{RHS} = C_+ C_- h_i \frac{d}{dx} A_i^2(x)\Big|_{x=\lambda^2}$$

$$\boxed{\text{thus} \quad e^{-\varepsilon(\theta)} \sim \frac{d}{dx} A_i^2(x)\Big|_{x=\lambda^2} \quad // \quad = -2\pi \frac{d}{dx} A_i^2(x)\Big|_{x \doteq \left(\frac{3}{4}e^{\theta}\right)^{\frac{2}{3}}}}$$

Observation

$W(x) =$ F-term potential $\qquad W_1(X) = \frac{X^3}{3} - X$ in the above case

Relation between" Wave function" and potential

$$A_i(t) = \int_{\mathcal{C}} e^{W_1\left(\frac{X}{\sqrt{t}}\right)(\sqrt{t})^3} dX$$

$\mathcal{N}=2$ SUSY perturbation $W(x)$

wave function $\varphi(t)$

Stokes multiplier (trivial)

" deformation" by $\hbar$

Stokes multiplier (nontrivial)

analytic solution (in CFT limit)

$\mathcal{N}=2$ TBA

New index

Any further generalizations?

$\mathcal{N}=2$ SUSY perturbation

other F term potential

$W(x)$

wave function $\varphi(t)$

Stokes multiplier (trivial)

" deformation" by $\hbar$

Stokes multiplier (nontrivial)

$\mathcal{N}=2$ TBA

New index

New

analytic solutions (in CFT limit)

?

N=2 minimal model : Least relevant perturbation (Cecotti et al)

$$W_k(X = 2\cos\theta) = \frac{2}{k+2}\cos(k+2)\theta$$

TBA remains almost the same (2 independent functions for $\forall k$)

$$2\mathcal{D}(\theta) = \varepsilon(\theta) + \int_{-\infty}^{\infty} \frac{\ln(1+\eta^2(\theta'))}{\mathrm{ch}(\theta-\theta')}\frac{d\theta'}{2\pi}$$

$$\eta(\theta) = \int \frac{e^{-\varepsilon(\theta')}}{\mathrm{ch}(\theta-\theta')}\frac{d\theta'}{2\pi}$$

$$2\mathcal{D}(\theta) = z\,\mathrm{ch}\,\theta - \ln 2\cos\frac{\pi}{k+2}$$

$$\Big\} (3,1)$$

Define wave function $\psi(t)$ by

$$\psi(t) = \int_{\mathcal{C}} e^{W_k\left(\frac{x}{\sqrt{t}}\right)(\sqrt{t})^{k+2}} dx$$

Fact

$\psi(t)$ solves $\left[-\frac{d^2}{dt^2} + t^k\right]\psi(t) = 0$

Explicit Solution

$$\psi(t) \sim \sqrt{t}\, K_{\frac{1}{k+2}}\left(\frac{2}{k+2} t^{\frac{k+2}{2}}\right)$$

Observation [Lukyanov, DDMST]

$$\left[-\frac{d^2}{dt^2}+(t^{2\alpha}-E)^k\right]\psi(t)=0$$

$\hookleftarrow$ ODE corresponding $SU(2)_k$ model $\quad \Omega = e^{\frac{2\pi i}{k+2}}$

Again $\alpha = 1/2$ case

T system $\quad T_j(E\Omega^{-1/2})T_j(E\Omega^{1/2}) = 1 + T_{j+1}(E)T_{j-1}(E)$

before h-deformation $\quad \Rightarrow \quad$ after h-deformation

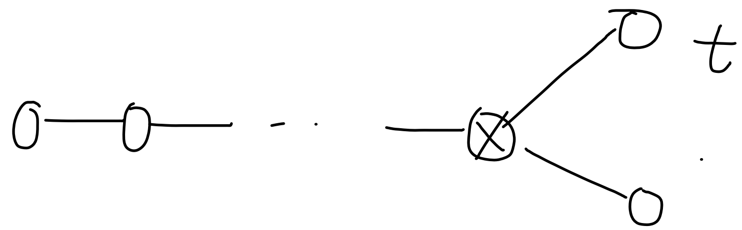
$$(T_j^{(0)})^2 = 1 + T_{j+1}^{(0)}T_{j-1}^{(0)}$$

T=numbers $\quad$ T = nontrivial functions

$$T_{k+1}^{(0)} = 0 \qquad T_k^{(0)} = 1$$

$$T_j^{(0)} = \sin\frac{(j+1)\pi}{k+2} \Big/ \sin\frac{\pi}{k+2}$$

After h-deformation



$$Y_j = T_{j+1} T_{j-1} \qquad \Omega^{\frac{k+1}{2}} = -1$$

$$Y_t = T_k \qquad \omega^{\frac{k+1}{2}} = -e^{-ih}$$

$$\left\{\begin{aligned}
&Y_j(E\Omega^{-1/2})\,Y_j(E\Omega^{1/2}) = (1+Y_{j-1})(1+Y_{j+1}) \qquad 1 \le j \le k-1 \quad (Y_0 = 0) \\
&Y_k(E\Omega^{-1/2})\,Y_k(E\Omega^{1/2}) = (1+Y_{k-1})(1+\omega^{\frac{k+2}{2}} Y_t)(1+\omega^{-\frac{k+2}{2}} Y_t) \\
&Y_t(E\Omega^{-1/2})\,Y_t(E\Omega^{1/2}) = 1+Y_k
\end{aligned}\right.$$

$\downarrow$

k+1 independent functions $\longleftrightarrow$ 2 independent functions in N=2 TBA

TBA decomposes into 2 parts

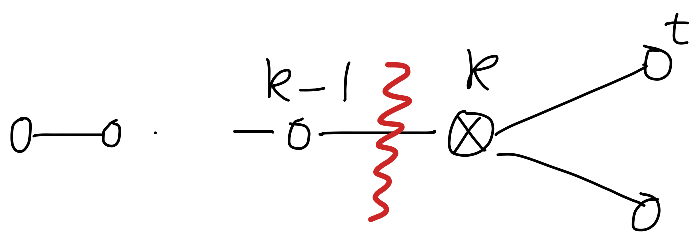
As $h \to 0$ reasonable to assume

$$T_j(E) = T_j^0 + h\, t_j(E) + O(h^2) \Rightarrow Y_j(E) = Y_j^0 + h\, y_j(E) + O(h^2)$$

$$Y_t(E) = T_k(E) = 1 + h\, y_t(E)$$

especially $T^0_{k+1} = 0 \Longrightarrow Y_k(E) = h\, y_k(E)$

then the lowest order equations decouple into 2-parts



$$\omega^{\frac{k+1}{2}} = -e^{-hi}$$

- $\underbrace{Y_k(E\Omega^{1/2})\, Y_k(E\Omega^{-1/2})}_{O(h^2)} = (1 + Y_{k-1})\underbrace{(1 + \omega^{\frac{k+1}{2}} Y_t)}_{O(h)}\underbrace{(1 + \omega^{-\frac{k+1}{2}} Y_t)}_{O(h)}$

  $\longrightarrow \; y_k(E\Omega^{1/2})\, y_k(E\Omega^{-1/2}) = \left[T^{(0)}_{k-1}\right]^2 (1 + y_t^2)$

- $Y_t(E\Omega^{1/2})\, Y_t(E\Omega^{-1/2})$

  $\longrightarrow \; y_t(E\Omega^{1/2}) + y_t(E\Omega^{-1/2}) = y_k(E)$

} recovers TBA by Cecotti et al.

By expanding the both sides of quantum Wronskian w.r.t. h

$\longrightarrow$ explicit solution

$$\begin{cases} y_k(E) \propto \varphi(E)\,\varphi'(E) \\ y_t(E) \propto \Omega^{-\frac{1}{2}}\varphi(E\Omega^{1/2})\,\varphi'(E\Omega^{-1/2}) \\ \qquad + \Omega^{1/2}\varphi(E\Omega^{-1/2})\,\varphi(E\Omega^{1/2}) \end{cases}$$

$$\varphi(t) \sim \sqrt{t}\, K_{\frac{1}{k+2}}\left(\frac{2}{k+2}\, t^{\frac{k+2}{2}}\right)$$

$$Q(z\to 0) = -2\times 8\cos\frac{\pi}{k+2}\int dE\, E^{k/2}\,\varphi(E)\,\varphi'(E) = \frac{k}{k+2}$$

$=$ agrees with Cecotti et al

§ Conclusion

- Another application of ODE/IM

  $\Rightarrow$ Find analytic solutions to TBA

(∞ -) Many open questions

- Why F-term potential W_R solves TBA?
- $\uparrow$ only $\mathbb{Z}_2$ case how about $\mathbb{Z}_n$?
- only $z = t\bar{t}^* \to 0$ limit : ODE/IM helpful for Z arbitrary ?
- Why ODE/IM ?

Appendix 1 Next leading perturbation (SU(2) case)

ex.) $W(x,t) = \frac{x^6}{6} - \frac{t x^2}{2}$

Cecotti et al ; redefinition of super field $x^2 \to X$

New index $Q \to 2Q$ of $W(X) = \frac{X^3}{3} - tX$

Observation in ODE/IM

$\psi(t) = \int_{\gamma} e^{W(x,t)} dx$ satisfies $\boxed{\frac{d^3}{dt^3}\psi = \frac{1}{4}\sqrt{t}\,\frac{d}{dt}\sqrt{t}\,\psi}$

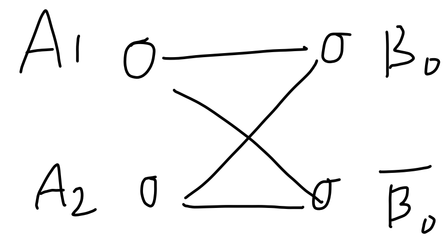
$\Rightarrow \quad \psi = Ai^2(ct) \,.\ Ai(ct)Bi(ct) \,.\ Bi^2(ct) \quad \left(c = 16^{\frac{1}{3}} \right)$

$Q = 2Q$ of $\frac{X^3}{3} - tX$ follows!

Appendix 2 $\mathbb{Z}_3$ case

$$W(x,t) = \frac{x^4}{4} - tX$$

Fendley-Intriligator TBA



observation from ODE/IM

- $\psi(t) = \int_{\mathcal{C}} e^{W(x,t)} dx$ satisfies $\boxed{\dfrac{d^3}{dt^3}\psi + t\psi = 0}$

- h deformation $\xrightarrow[h\to 0]{}$ Fendley-Intriligator TBA recovered

- $$\psi(t) \sim \lim_{N\to\infty} {}_3F_2\left(\begin{matrix} -N, & -N-\frac{1}{4} & -N-\frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} & \end{matrix} \,\middle|\, \frac{x^4}{N^3}\right)$$

Elementary ?