CORRELATION FUNCTIONS IN QUANTUM INTEGRABLE MODELS FROM Q-DEFORMATION OF CANONICAL NORMALISED SECOND KIND DIFFERENTIAL.

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1. From XIX century algebraic geometry.

Consider an algebraic curve

$$R(z,w)=0\,,$$

requiring that it is non-singular, i.e. there are no solutions to the equations:

$$R(z,w)=0, \quad \partial_z R(z,w)=0, \quad \partial_w R(z,w)=0\,.$$

Holomorphic differentials:

$$\sigma_j=\frac{P_j(z,w)}{\partial_w R(z,w)}dz\,,$$

where $P_j(z,w)$ are chosen requiring regularity at infinite points.
Near the branch point in z-plain (solution to
$R(z,w)=0, \quad \partial_w R(z,w)=0\,$.) we take w as variable:

$$\sigma_j=-\frac{P_j(z,w)}{\partial_z R(z,w)}dw\,.$$

There are g (algebraic genus) of such differentials.

Geometrical picture: Riemann surface of genus g. Chose the homology basis

$$(\alpha_1, \cdots, \alpha_g \,;\, \beta_1, \cdots, \beta_g)\,,$$

with canonical intersections:

$$\alpha_i \circ \alpha_j = 0, \quad \beta_i \circ \beta_j = 0, \quad \alpha_i \circ \beta_j = \delta_{i,j}\,.$$

Non-degeneracy condition 0.

For non-singular algebraic curve

$$\det\Big(\int_{\alpha_i} \sigma_j\Big) \neq 0\,.$$

This allows to introduce normalised 1-kind differentials:

$$\omega_j \ : \quad \int_{\alpha_i} \omega_j = \delta_{i,j}\,.$$

2-kind differentials have singularities, but no residues. Up to exact forms

$$dR(p)\,, \quad \text{from now on} \quad p=(z,w)\,,$$

there are g of them.

Useful exact form (with respect to p_1):

$$\rho_0(p_1,p_2)=\frac{dz_2}{\partial_{w_2}R(z_2,w_2)}d_1\left(\frac{R(z_1,w_2)}{(z_1-z_2)(w_1-w_2)}\right)\,.$$

We have for $p_1\to p_2$:

$$\rho_0(p_1,p_2)=\left(\frac{1}{(z_1-z_2)^2}+O(1)\right)dz_1dz_2\,.$$

On the other hand for $z_1\to\infty$

$$\rho_0(p_1,p_2)=\sum_{j=1}^{g}\sigma(p_2)\tilde{\sigma}(p_1)+O(1)\,,$$

Intorduce the two form

$$\sigma(p_1, p_2) = \rho_0(p_1, p_2) - \rho_0(p_2, p_1)\,.$$

It has no singularity at $p_1 = p_2$, hence

$$\sigma(p_1, p_2) = \sum_{j=1}^{g} \left(\sigma(p_2)\tilde{\sigma}(p_1) - \sigma(p_1)\tilde{\sigma}(p_2)\right)\ .$$

Differentials $\sigma_i, \tilde{\sigma}_i$ are defined up to the action of $\mathrm{Sp}(2g)$. This is not the modular group, but its close cousin. It can be shown that they are dual with respect to

$$\eta_1 \circ \eta_2 = \int_\gamma \eta_1 d^{-1} \eta_2\,,$$

where γ goes around $z = \infty$.

Riemann bilinear identity.

The form $\sigma(p_1, p_2)$ is exact, but singular. It is easy to see that

$$\frac{1}{2\pi i} \int_{\gamma_1} \int_{\gamma_2} \sigma(p_1, p_2) = \gamma_1 \circ \gamma_2 \, .$$

So, the matrix of periods for $\sigma_j, \tilde{\sigma}_j$ belongs to $\mathrm{Sp}(2g)$. The modular group $\mathrm{Sp}(2g, \mathbb{Z})$ acts on this matrix from cycle's side.
Multiplying in opposite direction we get more conventional form:

$$\frac{1}{2\pi i} \sum_{j=1}^{g} \Big(\int_{\alpha_j} \eta_1 \int_{\beta_j} \eta_2 - \int_{\alpha_j} \eta_2 \int_{\beta_j} \eta_1 \Big) = \eta_1 \circ \eta_2 \, .$$

Consider one quoter of RBI (1/4RBI):

$$\int_{\alpha_i}\int_{\alpha_j}\sigma(p_1,p_2)=0\,.$$

Introduce matrices of a-periods:

$$\mathcal{A}_{i,j}=\int_{\alpha_j}\sigma_j\,,\quad \mathcal{B}_{i,j}=\int_{\alpha_j}\tilde{\sigma}_j\,.$$

Due to 1/4RBI the matrix $\mathcal{X}=\mathcal{A}^{-1}\mathcal{B}$ is symmetric.

Canonical normalised second kind differential.

This is a fundamental for us object. Definition:

$$\rho(p_1,p_2)=\left(\frac{1}{(z_1-z_2)^2}+O(1)\right)dz_1dz_2,\qquad \int_{\alpha_j}\rho(p_1,p_2)=0\,,\ \forall j\,.$$

Construction:

$$\rho(p_1,p_2) = \rho_0(p_1,p_2) - \sum_{j=1}^{g} \sigma(p_2)\tilde{\sigma}(p_1) + \sum_{i,j=1}^{g} \mathcal{X}_{i,j}\sigma_j(p_1)\sigma_i(p_2)\,.$$

It is symmetric due to the fact that $\mathcal{X}$ is symmetric. We need only 1/4RBI to prove this.
One more property:

$$\int_{\beta_j} \rho(p_1,p_2) = \omega_j(p_2)\,.$$

Applications of the canonical normalised second kind differential.

Normalised differential with locally defined singular part $df(p)$ can be constructed using $\rho(p_1,p_2)$:

$$\eta(p) = \int_{\Gamma} \rho(p,p_1) f(p_1)\,.$$

Just for fun. Suppose $f(p)$ is globally defined. Then all periods of df vanish. In our construction a-periods vanish automatically, for b-periods we have a system of linear requirements:

$$\int_\Gamma \omega_j(p) f(p) = 0\,, \quad \forall j\,.$$

This implies Riemann-Roch theorem.

Important facts to remember from this part.

1. Non-degeneracy condition: determinant of a-periods of holomorphic differentials do not vanish.

2. Symmetry of the canonical 2-kind differential requires 1/4RBI.

3. Every normalised differential is constructed from its singular part via the canonical 2-kind differential.

2. Formulation of our problem

We define two spaces: $\mathfrak{H}_{\mathbf{S}} = \bigotimes_{j=-\infty}^{\infty} \mathbb{C}^2$, $\mathfrak{H}_{\mathbf{M}} = \bigotimes_{\mathbf{m}=\mathbf{1}}^{\mathbf{n}} \mathbb{C}^{d_{\mathbf{m}}}$, and the rectangular transfer-matrix:

$$T_{\mathrm{S},\mathbf{M}} = \prod_{j=-\infty}^{\overset{\curvearrowright}{\infty}} T_{j,\mathbf{M}} \, ,$$

where

$$T_{j,\mathbf{M}} \equiv T_{j,\mathbf{M}}(1), \quad T_{j,\mathbf{M}}(\zeta) = \prod_{\mathbf{m}=\mathbf{1}}^{\overset{\curvearrowleft}{\mathbf{n}}} L_{j,\mathbf{m}}(\zeta/\tau_{\mathbf{m}}) \, .$$

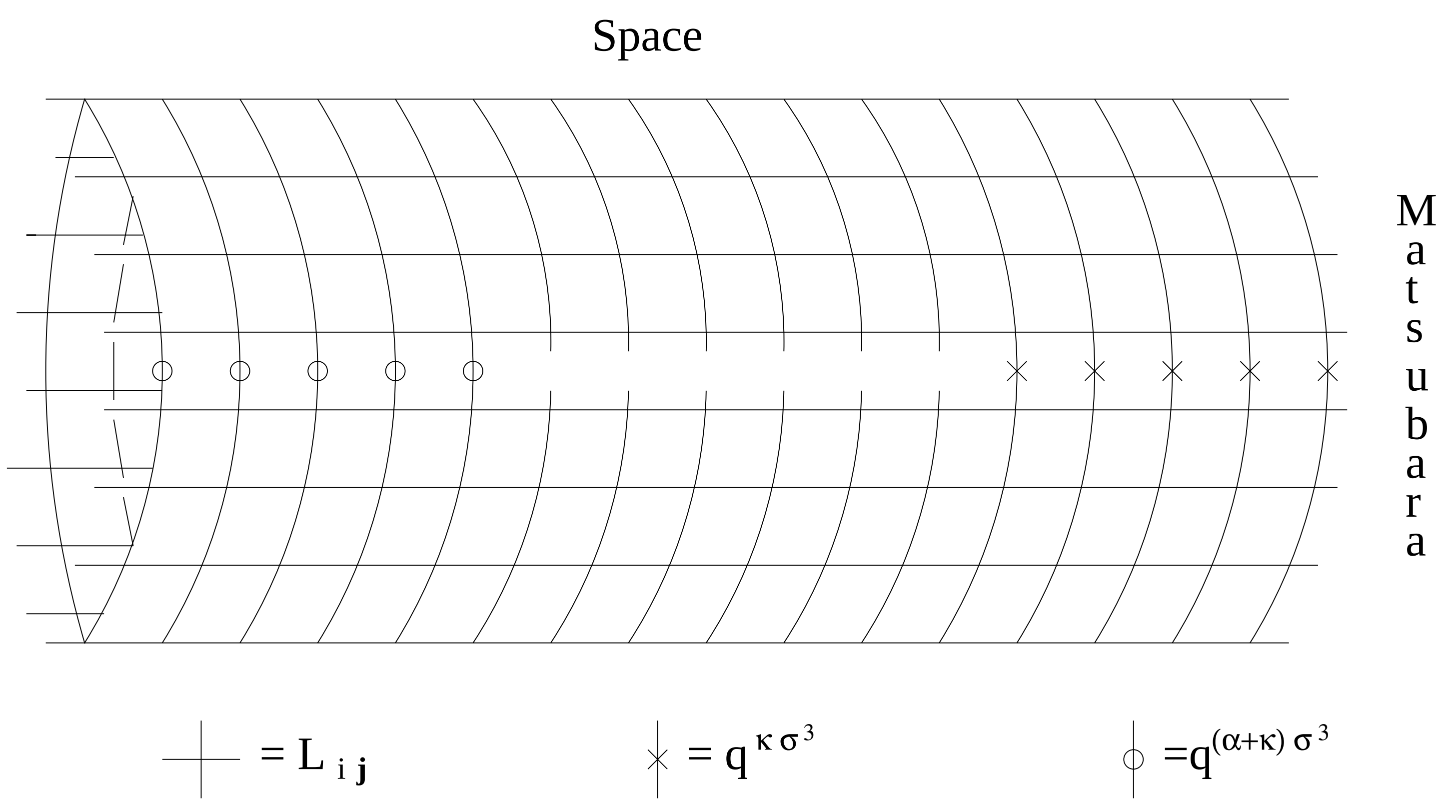
The L-operators are obtained form the universal one

$$L_j(\zeta) = q^{\frac{1}{2}} \begin{pmatrix} \zeta^2 q^{\frac{H+1}{2}} - q^{-\frac{H+1}{2}} & (q-q^{-1})\zeta F q^{\frac{H-1}{2}} \\ (q-q^{-1})\zeta q^{-\frac{H-1}{2}} E & \zeta^2 q^{-\frac{H-1}{2}} - q^{\frac{H-1}{2}} \end{pmatrix}_j \, ,$$

We consider the linear functional

$$Z^{\kappa}\left\{q^{2\alpha S(0)}\mathcal{O}\right\} = \frac{\mathrm{Tr}_{\mathbf{M}}\mathrm{Tr}_{\mathbf{S}}\left(T_{\mathbf{S},\mathbf{M}}\ q^{2(\kappa S+\alpha S(0))}\mathcal{O}\right)}{\mathrm{Tr}_{\mathbf{M}}\mathrm{Tr}_{\mathbf{S}}\left(T_{\mathbf{S},\mathbf{M}}\ q^{2(\kappa S+\alpha S(0))}\right)}\ ,$$

where $S=\frac{1}{2}\sum_{j=-\infty}^{\infty}\sigma_j^3$, $S(0)=\frac{1}{2}\sum_{j=-\infty}^{0}\sigma_j^3$, $\mathcal{O}$ is local. Graphically:



Evaluation of Z on descendant created by fermions. Consider two Matsubara transfer-matrices:

$$T_{\mathbf{M}}(\zeta,\kappa)=\mathrm{Tr}_j(T_{j,\mathbf{M}}(\zeta)q^{\kappa\sigma_j^3}),\ \ T_{\mathbf{M}}(\zeta,\kappa+\alpha)=\mathrm{Tr}_j(T_{j,\mathbf{M}}(\zeta)q^{(\kappa+\alpha)\sigma_j^3})\,.$$

Denote by $|\kappa\rangle$, $\langle\kappa+\alpha|$ the eigenvectors corresponding to maximal eigenvalues of $T_{\mathbf{M}}(1,\kappa)$ and $T_{\mathbf{M}}(1,\kappa+\alpha)$.
Suppose $q^{2\alpha S(0)}\mathcal{O}=q^{2\alpha S(k-1)}X_{[k,m]}$. It is easy to see that

$$Z^{\kappa}\Bigl\{q^{2\alpha S(k-1)}X_{[k,m]}\Bigr\}=\frac{T(1,\alpha+\kappa)^{k-1}}{T(1,\kappa)^m}\times\frac{\langle\kappa+\alpha|\mathrm{Tr}_{[k,m]}\left(T_{[k,m],\mathbf{M}}q^{2\kappa S_{[k,m]}}X_{[k,m]}\right)|\kappa\rangle}{\langle\kappa+\alpha|\kappa\rangle}\,.$$

We need

Non-degeneracy condition 1.

$$\langle\kappa+\alpha|\kappa\rangle\neq 0\,.$$

3. Analysis of Matsubara data. Deformed Abelian differentials.

In what follows one has to keep in mind an analogy with the analysis of hyper-elliptic Riemann surface. In the previous notations it corresponds to

$$R(z,w)=a(z)w^{-1}+d(z)w-T(z)=0\,,$$

with $\mathrm{deg}a(z)=\mathrm{deg}d(z)=\mathrm{deg}T(z)=\mathbf{n}$. Genus is $g=\mathbf{n}-1$.

Consider $T_{\mathbf{M}}(\zeta,\lambda)$ for us $\lambda=\kappa,\kappa+\alpha$. We have in addition two solutions to Baxter equation ($z=\zeta^2$, the notations are not perfect):

$$T_{\mathbf{M}}(\zeta,\lambda)Q^{\pm}_{\mathbf{M}}(\zeta,\lambda)=d(\zeta)Q^{\pm}_{\mathbf{M}}(\zeta q,\lambda)+a(\zeta)Q^{\pm}_{\mathbf{M}}(\zeta q^{-1},\lambda)\,,$$

where

$$a(\zeta)=\prod_{\mathbf{m}=1}^{\mathbf{n}}a_{s_{\mathbf{m}}}(\zeta/\tau_{\mathbf{m}}),\qquad a_s(\zeta)=\zeta^2q^{2s+1}-1\,,$$

$$d(\zeta)=\prod_{\mathbf{m}=1}^{\mathbf{n}}d_{s_{\mathbf{m}}}(\zeta/\tau_{\mathbf{m}}),\qquad d_s(\zeta)=\zeta^2q^{-2s+1}-1\,.$$

Two solutions are specified by

$$Q^{\pm}_{\mathbf{M}}(\zeta,\lambda)=\zeta^{\pm(\lambda-\mathbf{S})}\mathrm{Pol}(\zeta^2)\,.$$

Under the spin reversal:

$$T_{\mathbf{M}}(\zeta,\lambda)=J\;T_{\mathbf{M}}(\zeta,-\lambda)\;J\,,\quad Q^{-}_{\mathbf{M}}(\zeta,\lambda)=J\;Q^{+}_{\mathbf{M}}(\zeta,-\lambda)\;J\,.$$

Quantum Wronskian

$$Q^{+}_{\mathbf{M}}(\zeta,\lambda)Q^{-}_{\mathbf{M}}(\zeta q,\lambda)-Q^{-}_{\mathbf{M}}(\zeta,\lambda)Q^{+}_{\mathbf{M}}(\zeta q,\lambda)=\frac{1}{q^{\lambda-\mathbf{S}}-q^{-\lambda+\mathbf{S}}}W(\zeta),$$

where

$$W(\zeta)=\prod_{\mathbf{m}=\mathbf{1}}^{\mathbf{n}}w_{s_{\mathbf{m}}}(\zeta/\tau_{\mathbf{m}}),\quad w_s(\zeta)=\prod_{k=1}^{2s}(1-\zeta^2q^{2k-2s+1})\,.$$

We introduce the function

$$\varphi(\zeta)=\prod_{\mathbf{m}=\mathbf{1}}^{\mathbf{n}}\varphi_{s_{\mathbf{m}}}(\zeta/\tau_{\mathbf{m}})\,,\quad \varphi_s(\zeta)=\prod_{k=0}^{2s}\frac{1}{\zeta^2q^{-2s+2k+1}-1}\,,$$

which satisfies

$$a(\zeta q)\varphi(\zeta q)=d(\zeta)\varphi(\zeta)\,.$$

Conours $\Gamma_{\mathbf{m}}$ contain different series of its poles:

$$\Gamma_{\mathbf{m}}\ \supset\ \{\tau_{\mathbf{m}}^2q^{2s_{\mathbf{m}}-2k-1};\quad k=0,\cdots,2s_{\mathbf{m}}\}\,.$$

In addition we introduce Γ_0 going around $\zeta^2=0$.
Deformed Abelian integrals (a-cycles):

$$\int\limits_{\Gamma_{\mathbf{m}}}f^{\pm}(\zeta)Q^{\mp}(\zeta,\kappa+\alpha)Q^{\pm}(\zeta,\kappa)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2}\,,$$

where $\zeta^{\mp\alpha}f^{\pm}(\zeta)$ is a polynomial (or maybe rational) in ζ^2, in order that the integrand is single-valued. From now on we consider the eigenvalues on our favourites states $|\kappa\rangle,\ |\kappa+\alpha\rangle$.

We shall need two difference operators:

$$\Delta_\zeta f(\zeta) = f(\zeta q) - f(\zeta q^{-1})\,, \quad \delta_\zeta f(\zeta) = f(\zeta) - \rho(z) f(\zeta q^{-1})\,,$$

where

$$\rho(\zeta) = \frac{T(\zeta, \kappa+\alpha)}{T(\zeta,\kappa)}\,.$$

“Primitive function" Δ_ζ^{-1} is well defined on $\zeta^{\pm\alpha}\mathrm{Pol}(\zeta^2)$.
Simple identity for $\zeta^{\pm\alpha}\mathrm{Pol}(\zeta^2)$

$$\int\limits_{\Gamma_{\mathbf{m}}} T(\zeta,\kappa)\left(\delta_\zeta \Delta_\zeta^{-1} f^{\pm}(\zeta)\right) Q^{\mp}(\zeta,\kappa+\alpha) Q^{\pm}(\zeta,\kappa)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2}$$
$$= \int\limits_{\Gamma_{\mathbf{m}}} f^{\pm}(\zeta) d(\zeta) Q^{\mp}(\zeta,\kappa+\alpha) Q^{\pm}(\zeta q,\kappa)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2}\,,$$

and definition otherwise.

Returning to $\zeta^{\pm\alpha}\mathrm{Pol}(\zeta^2)$ we observe two fundamental facts.
q-deformed exact forms. Define a q-deformed exact form to be an expression

$$
\begin{aligned}
&E\bigl(f^{\pm}(\zeta)\bigr) \\
&= T(\zeta,\kappa)\Delta_\zeta^{-1}\left(f^{\pm}(\zeta)T(\zeta,\kappa)\right) + T(\zeta,\kappa+\alpha)\Delta_\zeta^{-1}\left(f^{\pm}(\zeta)T(\zeta,\kappa+\alpha)\right) \\
&- T(\zeta,\kappa)\Delta_\zeta^{-1}\left(f^{\pm}(\zeta q)T(\zeta q,\kappa+\alpha)\right) - T(\zeta,\kappa+\alpha)\Delta_\zeta^{-1}\left(f^{\pm}(\zeta q^{-1})T(\zeta q^{-1},\kappa)\right) \\
&+ a(\zeta q)d(\zeta)f^{\pm}(\zeta q) - d(\zeta q^{-1})a(\zeta)f^{\pm}(\zeta q^{-1})\,,
\end{aligned}
$$

where $f^{\pm}(\zeta)=\zeta^{\pm\alpha}\mathrm{Pol}(\zeta^2)$. Then we have

$$
\int\limits_{\Gamma_{\mathbf{m}}} E\bigl(f^{\pm}(\zeta)\bigr)Q^{\mp}(\zeta,\kappa+\alpha)Q^{\pm}(\zeta,\kappa)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2}=0\,.
$$

With this the degree of polynomial can be reduced to $2\mathbf{n}-1$, so, we have

$\mathbf{n}+\mathbf{n}$ different polynomials.

1/4 of Riemann bilinear identities.

Consider the following function in two variables

$$r(\zeta,\xi)=r^{+}(\zeta,\xi)-r^{-}(\xi,\zeta)\,,$$

where $\quad r^{+}(\zeta,\xi)=r^{+}(\zeta,\xi|\kappa,\alpha),\quad r^{-}(\xi,\zeta)=r^{+}(\xi,\zeta|-\kappa,-\alpha),$

and

$$\begin{aligned}
r^{+}(\zeta,\xi|\kappa,\alpha)&=T(\zeta,\kappa)\Delta_\zeta^{-1}\left(\psi(\zeta/\xi,\alpha)(T(\zeta,\kappa)-T(\xi,\kappa))\right)\\
&+T(\zeta,\kappa+\alpha)\Delta_\zeta^{-1}\left(\psi(\zeta/\xi,\alpha)(T(\zeta,\kappa+\alpha)-T(\xi,\kappa+\alpha))\right)\\
&-T(\zeta,\kappa)\Delta_\zeta^{-1}\left(\psi(q\zeta/\xi,\alpha)(T(\zeta q,\kappa+\alpha)-T(\xi,\kappa+\alpha))\right)\\
&-T(\zeta,\kappa+\alpha)\Delta_\zeta^{-1}\left(\psi(q^{-1}\zeta/\xi,\alpha)(T(\zeta q^{-1},\kappa)-T(\xi,\kappa))\right)\\
&+\big(a(\zeta q)-a(\xi)\big)d(\zeta)\psi(q\zeta/\xi,\alpha)-\big(d(\zeta q^{-1})-d(\xi)\big)a(\zeta)\psi(q^{-1}\zeta/\xi,\alpha)\,.
\end{aligned}$$

Then

$$\int\limits_{\Gamma_{\mathbf{i}}}\int\limits_{\Gamma_{\mathbf{j}}} r(\zeta,\xi)Q^{-}(\zeta,\kappa+\alpha)Q^{+}(\zeta,\kappa)Q^{+}(\xi,\kappa+\alpha)Q^{-}(\xi,\kappa)\varphi(\zeta)\varphi(\xi)\frac{d\zeta^2}{\zeta^2}\frac{d\xi^2}{\xi^2}=0\,.$$

Clearly $\xi^{\alpha} r^{+}(\zeta, \xi)$ is a polynomial in ξ^2 and $\zeta^{-\alpha} r^{-}(\xi, \zeta)$ is a polynomial in ζ^2, both of degree $\mathbf{n}$. This allows us to define the polynomials $p^{\pm}_{\mathbf{m}}$ by

$$r^{+}(\zeta, \xi) = \sum_{\mathbf{m}=0}^{\mathbf{n}} \zeta^{\alpha} p^{+}_{\mathbf{m}}(\zeta^2) \xi^{-\alpha+2\mathbf{m}}, \qquad r^{-}(\xi, \zeta) = \sum_{\mathbf{m}=0}^{\mathbf{n}} \xi^{-\alpha} p^{-}_{\mathbf{m}}(\xi^2) \zeta^{\alpha+2\mathbf{m}}.$$

Introduce the $(\mathbf{n}+\mathbf{1}) \times (\mathbf{n}+\mathbf{1})$ matrices

$$\mathcal{A}^{\pm}_{\mathbf{i},\mathbf{j}} = \int_{\Gamma_{\mathbf{i}}} \zeta^{\pm\alpha+2\mathbf{j}} Q^{\mp}(\zeta, \kappa+\alpha) Q^{\pm}(\zeta, \kappa) \varphi(\zeta) \frac{d\zeta^2}{\zeta^2},$$

$$\mathcal{B}^{\pm}_{\mathbf{i},\mathbf{j}} = \int_{\Gamma_{\mathbf{i}}} \zeta^{\pm\alpha} p^{\pm}_{\mathbf{j}}(\zeta^2) Q^{\mp}(\zeta, \kappa+\alpha) Q^{\pm}(\zeta, \kappa) \varphi(\zeta) \frac{d\zeta^2}{\zeta^2}.$$

Then 1/4RBI means

$$\mathcal{B}^{+}(\mathcal{A}^{-})^{t} = \mathcal{A}^{+}(\mathcal{B}^{-})^{t}.$$

4. Fermions.

On the space $\mathcal{W}^{(\alpha)} = \bigoplus_{s=-\infty}^{\infty} \mathcal{W}_{\alpha-s,s}$. we define action of fermions:

$$\mathbf{b}^*(\zeta) = \sum_{p=1}^{\infty} (\zeta^2-1)^{p-1}\mathbf{b}^*_p\,, \quad \mathbf{c}^*(\zeta) = \sum_{p=1}^{\infty} (\zeta^2-1)^{p-1}\mathbf{c}^*_p\,,$$

$$\mathbf{b}(\zeta) = \sum_{p=0}^{\infty} (\zeta^2-1)^{-p}\mathbf{b}_p\,, \quad \mathbf{c}(\zeta) = \sum_{p=0}^{\infty} (\zeta^2-1)^{-p}\mathbf{c}_p\,.$$

Non-trivial commutation relations are:

$$[\mathbf{b}(\zeta_1), \mathbf{b}^*(\zeta_2)]_+ = -\psi(\zeta_2/\zeta_1, \alpha)\,, \quad [\mathbf{c}(\zeta_1), \mathbf{c}^*(\zeta_2)]_+ = \psi(\zeta_1/\zeta_2, \alpha)\,,$$

where

$$\psi(\zeta, \alpha) = \zeta^\alpha \frac{\zeta^2+1}{2(\zeta^2-1)}\,.$$

Fermions have the block structure:

$$\mathbf{b}^*_p, \mathbf{c}_p \ : \ \mathcal{W}_{\alpha-s+1,s-1} \ \rightarrow \ \mathcal{W}_{\alpha-s,s}\,, \quad \mathbf{c}^*_p, \mathbf{b}_p \ : \ \mathcal{W}_{\alpha-s-1,s+1} \ \rightarrow \ \mathcal{W}_{\alpha-s,s}\,.$$

Annihilation operators kill the primary field: $\mathbf{c}_p(q^{2\alpha S(0)}) = \mathbf{b}_p(q^{2\alpha S(0)}) = 0$.

We have two lemmas whose proofs are purely algebraic.

Lemma 1.

We have

$$T(\zeta,\kappa)Z^{\kappa}\Big\{\Big(\mathbf{b}^*(\zeta,\alpha)-\frac{1}{2\pi i}\oint_{\Gamma}\omega_{sing}(\zeta,\xi)\mathbf{c}(\xi,\alpha)\frac{d\xi^2}{\xi^2}\Big)(X)\Big\}=\zeta^{\alpha}P_{\mathbf{n}}(\zeta^2),$$

where $X\in\mathcal{W}_{\alpha+1,-1}$, Γ *encircles* $\xi^2=1$, *and* $P_{\mathbf{n}}(\zeta^2)$ *is a polynomial in* ζ^2 *of degree* $\mathbf{n}$, *and*

$$\begin{aligned}\omega_{\rm sing}(\zeta,\xi)=&\frac{1}{4}\frac{1}{T(\zeta,\kappa)T(\xi,\kappa)}\big(a(\xi)d(\zeta)\psi(q\zeta/\xi,\alpha)-a(\zeta)d(\xi)\psi(q^{-1}\zeta/\xi,\alpha)\big)\\&+\delta_\zeta\delta_\xi\Delta_\zeta^{-1}\psi(\zeta/\xi,\alpha)\,.\end{aligned}$$

Lemma 2.

The a-periods vanish

$$\int\limits_{\Gamma_{\mathbf{m}}} T(\zeta,\kappa)Z^{\kappa}\{\mathbf{b}^*(\zeta,\alpha)(X)\}Q^{-}(\zeta,\kappa+\alpha)Q^{+}(\zeta,\kappa)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2}=0,$$

for $X \in \mathcal{W}_{\alpha+1,-1}$, $\mathbf{m}=\mathbf{0},\mathbf{1},\cdots,\mathbf{n}$.

Now we can follow our XIX century logic: $T(\zeta,\kappa)Z^{\kappa}\{\mathbf{b}^*(\zeta,\alpha)(X)\}$ is a normalised differential with singular part given by
$\frac{1}{2\pi i}\oint\limits_{\Gamma}\omega_{sing}(\zeta,\xi)Z^{\kappa}\{\mathbf{c}(\xi,\alpha)(X)\}\frac{d\xi^2}{\xi^2}$.
Hence

$$Z^{\kappa}\big\{\mathbf{b}^*(\zeta)(X)\big\}=\frac{1}{2\pi i}\oint\limits_{\Gamma}\omega(\zeta,\xi)Z^{\kappa}\big\{\mathbf{c}(\xi)(X)\big\}\frac{d\xi^2}{\xi^2}\,,$$

with Γ going around 1. It remains to describe the canonical 2-kind differential $\omega(\zeta,\xi)$.

From the properties above $\omega(\zeta,\xi)$ must satisfy the requirements:

1. Singular part:

$$T(\zeta,\kappa)\Big(\omega(\zeta,\xi)-\omega_{sing}(\zeta,\xi)\Big)=\zeta^{\alpha}P_{\mathbf{n}}(\zeta^2)\,.$$

2. Normalisation:

$$\int\limits_{\Gamma_{\mathbf{m}}} T(\zeta,\kappa)\omega(\zeta,\xi)Q^{-}(\zeta,\kappa+\alpha)Q^{+}(\zeta,\kappa)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2}=0,\quad \mathbf{m}=0,\cdots,\mathbf{n}\,.$$

It is more or less immediate to write a formula

$$\omega(\zeta,\xi)=\frac{1}{T(\zeta,\kappa)T(\xi,\kappa)}v^{+}(\zeta)^{t}(\mathcal{A}^{+})^{-1}\mathcal{B}^{+}v^{-}(\xi)+\omega_{sing}(\zeta,\xi)(\zeta,\xi)\,,$$

Writing explicitly the dependence on α and recalling 1/4RBI we obtain the symmetry:

$$\omega(\zeta,\xi|-\alpha)=\omega(\xi,\zeta|\alpha)\,.$$

Non-degeneracy condition 2.

$$\det \mathcal{A}^{+} \neq 0\,.$$

Recall that $\mathcal{A}^{+}$ is the matrix of a-periods:

$$\mathcal{A}^{+}_{\mathbf{i},\mathbf{j}} = \int\limits_{\Gamma_{\mathbf{i}}} \zeta^{\alpha+2\mathbf{j}} Q^{-}(\zeta,\kappa+\alpha) Q^{+}(\zeta,\kappa)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2}\,.$$

Its similarity with the Non-degeneracy condition 0 is transparent. However, we are mostly interested in its equivalence to the Non-degeneracy condition 1:

$$\langle \kappa+\alpha|\kappa\rangle \neq 0\,.$$

Usually one writes

$$\langle\kappa+\alpha|\kappa\rangle = \langle -|\prod B(\mu_j^-)\prod C(\lambda_j^-)|-\rangle\,,$$

where $(\mu_j^\pm)^2$ are zeros of $\zeta^{\mp\kappa}Q^\pm(\zeta,\kappa+\alpha)$ and $(\lambda_j^\pm)^2$ are zeros of $\zeta^{\mp\kappa}Q^\pm(\zeta,\kappa)$. There is another way to proceed:

$$\langle\kappa+\alpha|\kappa\rangle = \langle -|\prod B(\mu_j^-)\prod B(\lambda_j^+)|+\rangle\,.$$

This is the domain wall partition function, which we normalise as

$$M_{\mathbf{n}}(\xi_{\mathbf{1}},\cdots,\xi_{\mathbf{n}}|\tau_{\mathbf{1}},\cdots,\tau_{\mathbf{n}}) = \prod \xi_{\mathbf{j}}^{-1}\langle -|\prod_{\mathbf{j}=1}^{\mathbf{n}} B(\xi_{\mathbf{j}})\,|+\rangle$$

with specification $\{\xi_{\mathbf{j}}\} = \{\mu_j^-\}\cup\{\lambda_j^+\}$. Verifying known recurrence relations we find:

$$\begin{aligned}&M_{\mathbf{n}}(\xi_{\mathbf{1}},\cdots,\xi_{\mathbf{n}}|\tau_{\mathbf{1}},\cdots,\tau_{\mathbf{n}})\\ &\quad = (-1)^{\mathbf{n}(\mathbf{n}-\mathbf{1})/2}\prod \tau_{\mathbf{j}}^{-2}\ \prod_{\mathbf{i},\mathbf{j}}(q\tau_{\mathbf{i}}^2 - q^{-1}\tau_{\mathbf{j}}^2)\prod_{\mathbf{i}<\mathbf{j}}(\tau_{\mathbf{i}}^2-\tau_{\mathbf{j}}^2)\det(\mathcal{A}^+)\,.\end{aligned}$$

5. Conclusion

Let us return to the formula

$$Z^\kappa\left\{q^{2\alpha S(k-1)}X_{[k,m]}\right\} = \frac{T(1,\alpha+\kappa)^{k-1}}{T(1,\kappa)^m} \times \frac{\langle\kappa+\alpha|\mathrm{Tr}_{[k,m]}\left(T_{[k,m],\mathbf{M}}q^{2\kappa S_{[k,m]}}X_{[k,m]}\right)|\kappa\rangle}{\langle\kappa+\alpha|\kappa\rangle}\,.$$

If $X = E_{\epsilon_k}^{\epsilon'_k}\cdots E_{\epsilon_m}^{\epsilon'_m}$ then the essential part

$$\begin{aligned}&\langle\kappa+\alpha|\mathrm{Tr}_{[k,m]}\left(T_{[k,m],\mathbf{M}}q^{2\kappa S_{[k,m]}}X_{[k,m]}\right)|\kappa\rangle\\ &=\langle\kappa+\alpha|T_{\mathbf{M}}(1,\kappa)_{\epsilon'_k}^{\epsilon_k}\cdots T_{\mathbf{M}}(1,\kappa)_{\epsilon'_m}^{\epsilon_m}|\kappa\rangle\,,\end{aligned}$$

where $T_{\mathbf{M}}(1,\kappa)_\epsilon^{\epsilon'}$ are matrix element (with respect to a) of $T_{a,\mathbf{M}}(\zeta)q^{\kappa\sigma_a^3}$. There are other ways to compute this. Why should we care about our fermions?

Because fermions provide nice linear combinations of operators for which the matrix element is simple. Moreover, they provide families of operators of different length with universally described value of Z^κ. The simplest example is

$$\mathbf{b}^*(\zeta)\mathbf{c}^*(\xi)\big(q^{2\alpha S(0)}\big) = \sum_{r,s=1}^{\infty}(\zeta^2-1)^{r-1}(\xi^2-1)^{s-1}X_{[1,r+s]}q^{2\alpha S(0)}\,,$$

where $X_{[1,r+s]}$ has support $[1,r+s]$. At the same time

$$Z^\kappa\Big\{\mathbf{b}^*(\zeta)\mathbf{c}^*(\xi)\big(q^{2\alpha S(0)}\big)\Big\} = \omega(\zeta,\xi)\,.$$

This gives the opportunity of scaling limit.