Spin chains based on (quantum) gl(3) algebras: Bethe vectors, scalar products and form factors

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General background: Integrable spin chains

Rational or trigonometric 9×9 R-matrix

$R(x,y) \in V \otimes V$ with $V = End(\mathbb{C}^3)$

$R(x,y)$ obeys Yang-Baxter equation (in $V \otimes V \otimes V$)

$$R^{12}(x_1,x_2)\, R^{13}(x_1,x_3)\, R^{23}(x_2,x_3) = R^{23}(x_2,x_3)\, R^{13}(x_1,x_3)\, R^{12}(x_1,x_2)$$

It is associated to a quantum group $\mathcal{A}$ which is:

- The Yangian $\mathcal{A} = Y(gl_3)$ when $R(x,y)$ is rational (XXX chain)
- The affine quantum group $\mathcal{A} = U_q(\widehat{gl}_3)$ when $R(x,y)$ is trigonometric (XXZ chain)

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For the talk, rational R-matrix:

$$R(x,y) = \mathbf{I} + g(x,y)\mathbf{P} \in End(\mathbb{C}^3) \otimes End(\mathbb{C}^3) \quad \text{and} \quad g(x,y) = \frac{c}{x-y}$$

$\mathbf{I}$ is the identity matrix, $\mathbf{P}$ is the permutation matrix between two spaces $End(\mathbb{C}^3)$, c is a constant.

Monodromy matrix

$$T(x) = \sum_{i,j=1}^{3} e_{ij} \otimes T_{ij}(x) \in \mathrm{End}(\mathbb{C}^3) \otimes \mathcal{A}$$

$T(x)$ obeys the RTT commutation relations:

$$R^{12}(x,y)\, T^1(x)\, T^2(y) = T^2(y)\, T^1(x)\, R^{12}(x,y)$$

This defines the quantum group $\mathcal{A}$
It leads to an integrable model through the transfer matrix

$$t(x) = tr_0\, T^0(x) = T_{11}(x) + T_{22}(x) + T_{33}(x) \in \mathcal{A}$$
$$[t(x)\,,\, t(y)] = 0$$

Choice of a $Y(gl_3)$ (lowest weight) representation:

$$T_{jj}(w)|0\rangle = \lambda_j(w)|0\rangle,\ j=1,2,3 \qquad T_{ij}(w)|0\rangle = 0, \qquad 1 \le j < i \le 3$$

Up to normalisation $T(w) \to \lambda_2^{-1}(w)T(w)$, only need the ratios

$$r_1(w) = \frac{\lambda_1(w)}{\lambda_2(w)}, \qquad r_3(w) = \frac{\lambda_3(w)}{\lambda_2(w)}.$$

where r_1 and r_3 are free functional parameters.

Aim

Compute the correlation functions $< \mathcal{O}_1 \cdots \mathcal{O}_n >= tr\Big(\mathcal{O}_1 \cdots \mathcal{O}_n\Big)$ for some local operators $\mathcal{O}_1, \cdots, \mathcal{O}_n$

If one has a basis of the space of states $\mathcal{H}$, $\{|\psi >\}$, then it is enough to compute $< \psi'|\mathcal{O}_1 \cdots \mathcal{O}_n|\psi >$
Since we have a basis $\mathcal{O}|\psi >= \sum < \psi'|\mathcal{O}|\psi > \ |\psi' >$, and we need "only" $< \psi|\psi' >$

Aim

- Compute Bethe vectors (BVs), eigenvectors of $t(x)$:

$$t(x)\,\mathbb{B}^{a,b}(\bar{u},\bar{v}) = \tau(x|\bar{u},\bar{v})\,\mathbb{B}^{a,b}(\bar{u},\bar{v}) \;\Rightarrow\; \text{Bethe ansatz eqs (BAE)}$$

- Action of $T_{ij}(\bar{x})$ on $\mathbb{B}^{a,b}(\bar{u},\bar{v})$
- Scalar product of off-shell BVs (without BAE)
- Form factors $\mathbb{C}^{a,b}(\bar{t},\bar{s})\,T_{ij}(\bar{x})\mathbb{B}^{a,b}(\bar{u},\bar{v})$

Plan of the talk

- Bethe vectors (BVs)
- Multiple actions of Yangian generators on BVs
- Scalar products of BVs
- Form factors and correlation functions
- Conclusion

Calculations are rather technical $\Rightarrow$ results only!

Presentation for $Y(gl_3)$ but most of the results are valid for $U_q(gl_3)$ (see at the end)

Notations

Apart from the functions $g(x,y) = \frac{c}{x-y}$, $r_1(x)$ and $r_3(x)$ we introduce

$$f(x,y) = \frac{x-y+c}{x-y}, \quad h(x,y) = \frac{f(x,y)}{g(x,y)}, \quad t(x,y) = \frac{g(x,y)}{h(x,y)}.$$

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- "bar" always denote sets of variables: $\bar{w}$, $\bar{u}$, $\bar{v}$ etc..
- $|.|$ is the dimension of a set: $\bar{w} = \{w_1, w_2\} \;\Rightarrow\; |\bar{w}| = 2$, etc...
- Individual elements of the sets have latin subscripts: w_j, u_k, etc..
- Subsets of variables are denoted by roman indices: $\bar{u}_{\rm I}$, $\bar{v}_{\rm iv}$, $\bar{w}_{\rm II}$, etc.
- Special case: $\bar{u}_j = \bar{u} \setminus \{u_j\}$, $\bar{w}_k = \bar{w} \setminus \{w_k\}$, etc...

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Shorthand notations for products of scalar functions:

$$f(\bar{u}_{\rm II}, \bar{u}_{\rm I}) = \prod_{u_j \in \bar{u}_{\rm II}} \prod_{u_k \in \bar{u}_{\rm I}} f(u_j, u_k),$$

$$r_1(\bar{u}_{\rm II}) = \prod_{u_j \in \bar{u}_{\rm II}} r_1(u_j); \quad g(v_k, \bar{w}) = \prod_{w_j \in \bar{w}} g(v_k, w_j), \quad etc..$$

Bethe vectors

Framework: Algebraic-Nested Bethe ansatz (Leningrad school 80's) [Faddeev, Kulish, Reshetikhin, Sklyanin, Takhtajan]

On-shell Bethe vectors

$$t(x)\,\mathbb{B}^{a,b}(\bar{u};\bar{v}) = \tau(x|\bar{u};\bar{v})\,\mathbb{B}^{a,b}(\bar{u};\bar{v})$$

$\bar{u} = \{u_1, \dots, u_a\}$ and $\bar{v} = \{v_1, \dots, v_b\}$ are the Bethe parameters.
$t(x)$-eigenvectors provided $\bar{u}$ and $\bar{v}$ obey the Bethe equations (BAEs):

$$\begin{aligned} r_1(\bar{u}_{\rm I}) &= \frac{f(\bar{u}_{\rm I},\bar{u}_{\rm II})}{f(\bar{u}_{\rm II},\bar{u}_{\rm I})} f(\bar{v},\bar{u}_{\rm I}), \\ r_3(\bar{v}_{\rm I}) &= \frac{f(\bar{v}_{\rm II},\bar{v}_{\rm I})}{f(\bar{v}_{\rm I},\bar{v}_{\rm II})} f(\bar{v}_{\rm I},\bar{u}). \end{aligned}$$

that hold for arbitrary partitions of the sets $\bar{u}$ and $\bar{v}$ into subsets $\{\bar{u}_{\rm I},\ \bar{u}_{\rm II}\}$ and $\{\bar{v}_{\rm I},\ \bar{v}_{\rm II}\}$.

Known formulas: Trace formula ['07 Tarasov & Varchenko]

$$\mathbb{B}^{a,b}(\bar{u};\bar{v}) = tr\Big(\mathbb{T}(\bar{u};\bar{v})\,\mathbb{R}(\bar{u};\bar{v})\, e_{21}^{\otimes a} \otimes\; e_{32}^{\otimes b}\Big) \in Y(gl_3)$$

where $\mathbb{T}$ is some product of $T(x)$'s and $\mathbb{R}$ of R-matrices.

Recursion formulas

$$\lambda_2(u_k) f(\bar{v}, u_k)\mathbb{B}^{a+1,b}(\bar{u};\bar{v}) = T_{12}(u_k)\mathbb{B}^{a,b}(\bar{u}_k;\bar{v}) + \\ + \sum_{i=1}^{b} g(v_i, u_k) f(\bar{v}_i, v_i)\, T_{13}(u_k)\mathbb{B}^{a,b-1}(\bar{u}_k;\bar{v}_i),$$

$$\lambda_2(v_k) f(v_k, \bar{u})\mathbb{B}^{a,b+1}(\bar{u};\bar{v}) = T_{23}(v_k)\mathbb{B}^{a,b}(\bar{u};\bar{v}_k) + \\ + \sum_{j=1}^{a} g(v_k, u_j) f(u_j, \bar{u}_j)\, T_{13}(v_k)\mathbb{B}^{a-1,b}(\bar{u}_j;\bar{v}_k).$$

Explicit formulas

$$\mathbb{B}^{a,b}(\bar{u};\bar{v}) = \sum \frac{\mathrm{K}_k(\bar{v}_{\mathrm{I}}|\bar{u}_{\mathrm{I}})}{\lambda_2(\bar{v}_{\mathrm{II}})\lambda_2(\bar{u})} \frac{f(\bar{v}_{\mathrm{II}},\bar{v}_{\mathrm{I}})f(\bar{u}_{\mathrm{II}},\bar{u}_{\mathrm{I}})}{f(\bar{v}_{\mathrm{II}},\bar{u})f(\bar{v}_{\mathrm{I}},\bar{u}_{\mathrm{I}})} T_{12}(\bar{u}_{\mathrm{II}})T_{13}(\bar{u}_{\mathrm{I}})T_{23}(\bar{v}_{\mathrm{II}})|0\rangle$$

Plus others with different order of T_{12}, T_{13}, T_{23}

The sums are taken over partitions of the sets $\bar{u} \Rightarrow \{\bar{u}_{\mathrm{I}},\bar{u}_{\mathrm{II}}\}$ and $\bar{v} \Rightarrow \{\bar{v}_{\mathrm{I}},\ \bar{v}_{\mathrm{II}}\}$ with $0 \le |\bar{u}_{\mathrm{I}}| = |\bar{v}_{\mathrm{I}}| = k \le \min(a,b)$.

$\mathrm{K}_k(\bar{v}_{\mathrm{I}}|\bar{u}_{\mathrm{I}})$ is the Izergin–Korepin determinant

$$\mathrm{K}_k(\bar{x}|\bar{y}) = \prod_{\ell<m}^{k} g(x_\ell,x_m)g(y_m,y_\ell)\cdot h(\bar{x},\bar{y})\ \det_k\left[t(x_i,y_j)\right].$$

All these formulas are related

- Explicit expressions obey the recursion formulas
- Trace formula obeys the recursion formulas
- Recursion formulas uniquely fix the BVs, once $\mathbb{B}^{a,0}(\bar{u},.)$ or $\mathbb{B}^{0,b}(.,\bar{v})$ are known.

Bethe vectors $\mathbb{B}^{a,b}(\bar{u};\bar{v})$, $|\bar{u}|=a$, $|\bar{v}|=b$

- On-shell BVs: $\bar{u},\bar{v}$ obey BAEs so that

$$t(x)\,\mathbb{B}^{a,b}(\bar{u};\bar{v})=\tau(x|\bar{u};\bar{v})\,\mathbb{B}^{a,b}(\bar{u};\bar{v})$$

- Off-shell BVs: $\bar{u},\bar{v}$ are left free

Dual Bethe vectors $\mathbb{C}^{a,b}(\bar{u};\bar{v})$, $|\bar{u}|=a$, $|\bar{v}|=b$

- On-shell dual BVs: $\bar{u},\bar{v}$ obey BAEs so that

$$\mathbb{C}^{a,b}(\bar{u};\bar{v})\,t(x)=\tau(x|\bar{u};\bar{v})\,\mathbb{C}^{a,b}(\bar{u};\bar{v})$$

- Off-shell dual BVs: $\bar{u},\bar{v}$ are left free

Correlation functions

How to compute $\mathcal{O}_{\mathbb{C},\mathbb{B}} = \langle \mathbb{C}|\mathcal{O}|\mathbb{B}\rangle$?

If $|\mathbb{B}\rangle$ is a complete basis (of transfer matrix eigenvectors), then

$$\mathcal{O}|\mathbb{B}\rangle = \sum_{\mathbb{B}'} \mathbb{O}_{\mathbb{B}\mathbb{B}'}|\mathbb{B}'\rangle \qquad (1)$$

$\rightarrow$ what is needed is $\langle \mathbb{C}|\mathbb{B}'\rangle$ and (1)

Local operators: $\mathcal{O} = \sum_{\ell=1}^{L}\sum_{i,j=1}^{3} \mathcal{O}_{ij}^{(\ell)}\, e_{ij}^{\ell} \;\Rightarrow\; \langle \mathbb{C}|e_{ij}^{\ell}|\mathbb{B}\rangle$

Further simplification: QISM

Expression of e_{ij}^{ℓ}, $i,j = 1,2,3$ and $\ell = 1,\dots L$, in terms of monodromy entries $T_{kl}(x)$ ['00 Maillet & Terras]:

$$e_{ij}^{\ell} = \big(t(0)\big)^{\ell-1}\, T_{ij}(0)\,\big(t(0)\big)^{-\ell}$$

$\Rightarrow$ we need "only" $T_{kl}(x)\mathbb{B}^{a,b}(\bar{u};\bar{v})$ and $\mathbb{C}^{a,b}(\bar{w};\bar{z})\mathbb{B}^{a,b}(\bar{u};\bar{v})$

Multiple actions of $T_{ij}(\bar{x})$ on $\mathbb{B}^{a,b}(\bar{u};\bar{v})$

$$|\bar{x}| = n\,, \quad \{\bar{u},\bar{x}\} = \bar{\eta}\,, \quad |\bar{\eta}| = a+n\,; \quad \{\bar{v},\bar{x}\} = \bar{\xi}\,, \quad |\bar{\xi}| = b+n$$

$$T_{13}(\bar{x})\mathbb{B}^{a,b}(\bar{u};\bar{v}) = \lambda_2(\bar{x})\,\mathbb{B}^{a+n,b+n}(\bar{\eta};\bar{\xi}).$$

$$T_{12}(\bar{x})\mathbb{B}^{a,b}(\bar{u};\bar{v}) = (-1)^n\lambda_2(\bar{x})\sum f(\bar{\xi}_{\mathrm{II}},\bar{\xi}_{\mathrm{I}})\mathsf{K}_n(\bar{\xi}_{\mathrm{I}}|\bar{x}+c)\,\mathbb{B}^{a+n,b}(\bar{\eta};\bar{\xi}_{\mathrm{II}}).$$

Sum on partitions $\bar{\xi} = \{\bar{\xi}_{\mathrm{I}};\bar{\xi}_{\mathrm{II}}\}$ with $|\bar{\xi}_{\mathrm{I}}| = n$

$$T_{23}(\bar{x})\mathbb{B}^{a,b}(\bar{u};\bar{v}) = (-1)^n\lambda_2(\bar{x})\sum f(\bar{\eta}_{\mathrm{I}},\bar{\eta}_{\mathrm{II}})\mathsf{K}_n(\bar{x}|\bar{\eta}_{\mathrm{I}}+c)\,\mathbb{B}^{a,b+n}(\bar{\eta}_{\mathrm{II}};\bar{\xi}).$$

Sum on partitions $\bar{\eta} = \{\bar{\eta}_{\mathrm{I}};\bar{\eta}_{\mathrm{II}}\}$ with $|\bar{\eta}_{\mathrm{I}}| = n$

Imply recursion relations as a subcase (n=1)
Similar expressions for any $T_{ij}(\bar{x})$ and for dual BVs

Scalar products of BVs

$$\mathcal{S}_{a,b} \equiv \mathcal{S}_{a,b}(\bar{u}^C, \bar{u}^B | \bar{v}^C, \bar{v}^B) = \mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C)\, \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B)$$

Superscripts ^{B} and ^{C} to denote *different* sets of parameters!

General formula given by Reshetikhin

$$\begin{aligned}\mathcal{S}_{a,b} = \sum & r_1(\bar{u}^B_{\mathrm{I}}) r_1(\bar{u}^C_{\mathrm{II}}) r_3(\bar{v}^B_{\mathrm{I}}) r_3(\bar{v}^C_{\mathrm{II}}) \\ & \times f(\bar{u}^C_{\mathrm{I}}, \bar{u}^C_{\mathrm{II}}) f(\bar{u}^B_{\mathrm{II}}, \bar{u}^B_{\mathrm{I}}) f(\bar{v}^C_{\mathrm{II}}, \bar{v}^C_{\mathrm{I}}) f(\bar{v}^B_{\mathrm{I}}, \bar{v}^B_{\mathrm{II}}) f(\bar{v}^C_{\mathrm{I}}, \bar{u}^C_{\mathrm{I}}) f(\bar{v}^B_{\mathrm{II}}, \bar{u}^B_{\mathrm{II}}) \\ & \times Z_{a-k,n}(\bar{u}^C_{\mathrm{II}}; \bar{u}^B_{\mathrm{II}} | \bar{v}^C_{\mathrm{I}}; \bar{v}^B_{\mathrm{I}}) Z_{k,b-n}(\bar{u}^B_{\mathrm{I}}; \bar{u}^C_{\mathrm{I}} | \bar{v}^B_{\mathrm{II}}; \bar{v}^C_{\mathrm{II}})\end{aligned}$$

$\bar{u}^B = \{\bar{u}^B_{\mathrm{I}}, \bar{u}^B_{\mathrm{II}}\}$, $\bar{u}^C = \{\bar{u}^C_{\mathrm{I}}, \bar{u}^C_{\mathrm{II}}\}$ with $|\bar{u}^B_{\mathrm{I}}| = |\bar{u}^C_{\mathrm{I}}| = k$ for $k = 0, \dots, a$
$\bar{v}^B = \{\bar{v}^B_{\mathrm{I}}, \bar{v}^B_{\mathrm{II}}\}$, $\bar{v}^C = \{\bar{v}^C_{\mathrm{I}}, \bar{v}^C_{\mathrm{II}}\}$ with $|\bar{v}^B_{\mathrm{I}}| = |\bar{v}^C_{\mathrm{I}}| = n$ for $n = 0, \dots, b$.

$Z_{a,b}$ so-called highest coefficient

$$Z_{a,b}(\bar{t}; \bar{x} | \bar{s}; \bar{y}) = (-1)^b \sum K_b(\bar{s} - c | \bar{w}_{\mathrm{I}}) K_a(\bar{w}_{\mathrm{II}} | \bar{t}) K_b(\bar{y} | \bar{w}_{\mathrm{I}}) f(\bar{w}_{\mathrm{I}}, \bar{w}_{\mathrm{II}}).$$

But $\mathcal{S}_{a,b}$ difficult to handle.....

Here we consider the scalar product of an on-shell Bethe vector

$$t(x)\,\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B) = \tau(x|\bar{u}^B,\bar{v}^B)\,\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B) \quad \text{and BAEs}$$

with a *twisted* dual on-shell Bethe vector

$$\mathbb{C}_\kappa^{a,b}(\bar{u}^C;\bar{v}^C)\,t_\kappa(x) = \tau_\kappa(x|\bar{u}^C,\bar{v}^C)\,\mathbb{C}_\kappa^{a,b}(\bar{u}^C;\bar{v}^C)$$

with twisted BAEs

$$t_\kappa(x) = tr\big(M\,T(x)\big) \quad \text{with} \quad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$t_\kappa(x) = T_{11}(x) + \kappa\,T_{22}(x) + T_{33}(x)$$

$$\mathcal{S}_{a,b} \equiv \mathcal{S}_{a,b}(\bar{u}^C,\bar{u}^B|\bar{v}^C,\bar{v}^B) = \mathbb{C}_\kappa^{a,b}(\bar{u}^C;\bar{v}^C)\,\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B)$$

$$\mathcal{S}_{a,b} = f(\bar{v}^C, \bar{u}^C) f(\bar{v}^B, \bar{u}^B) t(\bar{v}^C, \bar{u}^B) \Delta'_a(\bar{u}^C) \Delta_a(\bar{u}^B) \Delta'_b(\bar{v}^C) \Delta_b(\bar{v}^B) \times \det_{a+b} \mathcal{M},$$

$$\Delta'_n(\bar{x}) = \prod_{j>k}^{n} g(x_j, x_k), \qquad \Delta_n(\bar{y}) = \prod_{j<k}^{n} g(y_j, y_k).$$

$\mathcal{M}$ is a $(a+b)\times(a+b)$ matrix. For $\bar{y} = \{\bar{u}^B, \bar{v}^C\}$:

$$\mathcal{M}_{j,k} = \frac{c}{g(y_k, \bar{u}^C) g(\bar{v}^C, y_k)} \frac{\partial \tau_\kappa(y_k|\bar{u}^C, \bar{v}^C)}{\partial u_j^C}, \quad j = 1, \dots, a,$$

$$\mathcal{M}_{a+j,k} = \frac{-c}{g(y_k, \bar{u}^B) g(\bar{v}^B, y_k)} \frac{\partial \tau(y_k|\bar{u}^B, \bar{v}^B)}{\partial v_j^B}, \quad j = 1, \dots, b.$$

Similar expression for $\mathcal{S}_{a,b}$ when considering a general twist

$$t_{\bar{\kappa}}(x) = \kappa_1 T_{11}(x) + \kappa_2 T_{22}(x) + \kappa_3 T_{33}(x)$$

but up to terms $(\kappa_i - 1)(\kappa_j - 1)$, $i, j = 1, 2, 3$

Form factors (off-diagonal case)

$$\mathcal{F}_{a,b}^{(i,j)}(z) = \mathbb{C}^{a',b'}(\bar{u}^C;\bar{v}^C)\,T_{ij}(z)\,\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B),$$
$$a' = a + \delta_{i1} - \delta_{j1}, \qquad b' = b + \delta_{j3} - \delta_{i3}, \quad i,j = 1,2,3.$$

Both $\mathbb{C}^{a',b'}(\bar{u}^C;\bar{v}^C)$ and $\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B)$ are on-shell Bethe vectors

$$\mathcal{F}_{a,b}^{(1,2)}(z) = \mathcal{H}_{a',b}\ \underset{a'+b}{\det}\,\mathcal{N}\,,$$

$$\mathcal{H}_{a',b} = \frac{\Delta'_{a'}(\bar{u}^C)\Delta'_b(\bar{v}^B)\Delta_{a+b+1}(\bar{x})}{h(\bar{v}^C,\bar{u}^B)}\,, \quad \bar{x} = \{\bar{u}^B,\bar{v}^C,z\}$$

$$\mathcal{N}_{j,k} = \frac{c}{g(x_k,\bar{u}^C)g(\bar{v}^C,x_k)}\frac{\partial\tau(x_k|\bar{u}^C,\bar{v}^C)}{\partial u_j^C}, \quad j = 1,\dots,a',$$

$$\mathcal{N}_{a'+j,k} = \frac{-c}{g(x_k,\bar{u}^B)g(\bar{v}^B,x_k)}\frac{\partial\tau(x_k|\bar{u}^B,\bar{v}^B)}{\partial v_j^B}, \quad j = 1,\dots,b.$$

Similar expression for all $\mathcal{F}_{a,b}^{(i,j)}(z)$, $|i-j| = 1$.

Form factors (diagonal case)

$$\mathcal{F}^{(s,s)}_{a,b}(z) = (-1)^b\, \mathcal{H}_{a',b} \cdot \det_{a+b+1} \mathcal{N}^{(s)},$$

$$\mathcal{N}^{(s)}_{j,k} = \mathcal{N}_{j,k}, \qquad j = 1, \dots, a+b, \qquad k = 1, \dots, a+b+1;$$

The last line of $\mathcal{N}^{(s)}$ depends on s, for instance:

$$\mathcal{N}^{(1)}_{a+b+1,k} = h(x_k, \bar{u}^B)h(\bar{v}^C, x_k)\left\{\frac{u_k^B}{c}\left(\frac{f(\bar{v}^B, u_k^B)}{f(\bar{v}^C, u_k^B)} - 1\right) - 1\right\},$$
$$k = 1, \dots, a;$$
$$\mathcal{N}^{(1)}_{a+b+1,a+k} = h(x_{a+k}, \bar{u}^B)h(\bar{v}^C, x_{a+k})\left\{\frac{v_k^C + c}{c}\left(\frac{f(v_k^C, \bar{u}^C)}{f(v_k^C, \bar{u}^B)} - 1\right) - 1\right\}$$
$$k = 1, \dots, b;$$
$$\mathcal{N}^{(1)}_{a+b+1,a+b+1} = \frac{r_1(z)f(\bar{u}^B, z)}{g(\bar{v}^C, z)g(z, \bar{u}^B)}$$

$\mathcal{N}^{(3)}$ has a similar expression;
$\mathcal{N}^{(2)}_{a+b+1,k} = h(x_k, \bar{u}^B)h(\bar{v}^C, x_k),\ k = 1, \dots, a+b+1.$

Form factors (remaining)

- $\mathcal{F}_{a,b}^{(i,j)}(z)$, $|i-j|=1$, are computed by brute force, using multiple action of T_{ij}'s
- $\mathcal{F}_{a,b}^{(s,s)}(z)$ are computed using the trick of "twisted BVs":

$$\mathcal{F}_{a,b}^{(s,s)}(z) = \frac{d}{d\kappa_s}\Big[\mathbb{C}_{\bar{\kappa}}^{a,b}(\bar{u}^C;\bar{v}^C)\big(t_{\bar{\kappa}}(z)-t(z)\big)\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B)\Big]_{\bar{\kappa}=1}$$

- $\mathcal{F}_{a,b}^{(1,3)}(z)$ and $\mathcal{F}_{a,b}^{(3,1)}(z)$ are not computable by these two methods: one needs a new one. We may have found a general one that could help to compute all the form factors in an easy way.
  More information soon...

Summary

For models with $GL(3)$ invariant R-matrix, we got:

- Explicit expressions for (off-shell) Bethe vectors and their duals
- Multiple action of monodromy elements on these BVs

Both results in term of Izergin-Korepin determinants
and sums of partitions of sets of Bethe parameters

- Calculation of the scalar product of (twisted) on-shell BVs
- Calculation of the form factors of $T_{ij}(x)$, $i, j = 1, 2, 3$

Both results in term of a single determinant
(and product of scalar functions)

Case of trigonometric R-matrices

- Explicit expression of BVs for $U_q(gl_N)$ arXiv:1310.3253
  Use of projectors method in current realization of $U_q(gl_N)$, see works of Khoroshkin, Pakuliak and collaborators
  arXiv:math/0610398, arXiv:math/0610433, arXiv:math/0610517, arXiv:0711.2819, arXiv:0810.3135, arXiv:1012.1455, etc...
- Multiple actions of $T_{ij}(\bar{w})$ for $U_q(gl_3)$ arXiv:1304.7602
- Scalar products in $U_q(gl_3)$: q-deformed Reshethekhin like formula
  arXiv:1311.3500, arXiv:1401.4355

Conclusion: still a lot to do...

- Calculation of the scalar product of generic off-shell BVs (as a single determinant)
  See for instance recent work of Wheeler, arXiv:1306.0552
- Complete calculation of correlation functions, asymptotics, etc...
- Generalization to other models
  - Calculation of the form factors of $T_{jk}(x)$, for $U_q(gl_3)$
  - Case of $U_q(gl_N)$ algebras

Thank you!

$\mathcal{F}^{(1,3)}_{a,b}(z)$ form factor

$$\begin{aligned}
\mathcal{F}^{(1,3)}_{a,b}(z) &= \mathbb{C}^{a+1,b+1}(\bar u^C;\bar v^C)\,T_{13}(z)\mathbb{B}^{a,b}(\bar u^B;\bar v^B)\\
&= (-1)^{b+1}\mathcal{H}_{a+1,b}\cdot \det_{a+b+2}\mathcal{N}^{(1,3)},\\
\mathcal{N}^{(1,3)}_{j,k} &= \mathcal{N}_{j,k}, \qquad j,k=1,\dots,a+b+1;\\
\mathcal{N}^{(1,3)}_{a+b+2,k} &= (-1)^{b+1}r_3(x_k)\frac{h(x_k,\bar v^B)}{g(x_k,\bar u^B)} + h(\bar v^B,x_k)h(x_k,\bar u^B).
\end{aligned}$$

Explicit formulas

$$\mathbb{B}^{a,b}(\bar{u};\bar{v}) = \sum \frac{K_k(\bar{v}_{\rm I}|\bar{u}_{\rm I})}{\lambda_2(\bar{v}_{\rm II})\lambda_2(\bar{u})} \frac{f(\bar{v}_{\rm II},\bar{v}_{\rm I})f(\bar{u}_{\rm II},\bar{u}_{\rm I})}{f(\bar{v}_{\rm II},\bar{u})f(\bar{v}_{\rm I},\bar{u}_{\rm I})} T_{12}(\bar{u}_{\rm II})T_{13}(\bar{u}_{\rm I})T_{23}(\bar{v}_{\rm II})|0\rangle$$

$$\mathbb{B}^{a,b}(\bar{u};\bar{v}) = \sum \frac{K_k(\bar{v}_{\rm I}|\bar{u}_{\rm I})}{\lambda_2(\bar{u}_{\rm II})\lambda_2(\bar{v})} \frac{f(\bar{v}_{\rm I},\bar{v}_{\rm II})f(\bar{u}_{\rm I},\bar{u}_{\rm II})}{f(\bar{v}_{\rm I},\bar{u}_{\rm I})f(\bar{v},\bar{u}_{\rm II})} T_{23}(\bar{v}_{\rm II})T_{13}(\bar{v}_{\rm I})T_{12}(\bar{u}_{\rm II})|0\rangle$$

$$\mathbb{B}^{a,b}(\bar{u};\bar{v}) = \sum \frac{K_k(\bar{v}_{\rm I}|\bar{u}_{\rm I})}{\lambda_2(\bar{v}_{\rm II})\lambda_2(\bar{u})} \frac{f(\bar{v}_{\rm II},\bar{v}_{\rm I})f(\bar{u}_{\rm I},\bar{u}_{\rm II})}{f(\bar{v},\bar{u})} T_{13}(\bar{u}_{\rm I})T_{12}(\bar{u}_{\rm II})T_{23}(\bar{v}_{\rm II})|0\rangle$$

$$\mathbb{B}^{a,b}(\bar{u};\bar{v}) = \sum \frac{K_k(\bar{v}_{\rm I}|\bar{u}_{\rm I})}{\lambda_2(\bar{u}_{\rm II})\lambda_2(\bar{v})} \frac{f(\bar{v}_{\rm II},\bar{v}_{\rm I})f(\bar{u}_{\rm I},\bar{u}_{\rm II})}{f(\bar{v},\bar{u})} T_{13}(\bar{v}_{\rm I})T_{23}(\bar{v}_{\rm II})T_{12}(\bar{u}_{\rm II})|0\rangle$$

The sums are taken over partitions of the sets $\bar{u} \Rightarrow \{\bar{u}_{\rm I},\bar{u}_{\rm II}\}$ and $\bar{v} \Rightarrow \{\bar{v}_{\rm I},\ \bar{v}_{\rm II}\}$ with $0 \le |\bar{u}_{\rm I}| = |\bar{v}_{\rm I}| = k \le \min(a,b)$.

$K_k(\bar{v}_{\rm I}|\bar{u}_{\rm I})$ is the Izergin–Korepin determinant

$$K_k(\bar{x}|\bar{y}) = \prod_{\ell<m}^{k} g(x_\ell,x_m)g(y_m,y_\ell)\cdot h(\bar{x},\bar{y})\ \det_k\left[t(x_i,y_j)\right].$$

The matrix $\mathcal{M}$

Diagonal blocks

$$
\begin{aligned}
\mathcal{M}^{(u)}(u_j^C, u_k^B) &= h(\bar{v}^C, u_k^B)h(u_k^B, \bar{u}^C)\Big[\kappa t(u_k^B, u_j^C) \\
&\quad + t(u_j^C, u_k^B)\frac{f(\bar{v}^B, u_k^B)}{f(\bar{v}^C, u_k^B)}\frac{h(\bar{u}^C, u_k^B)h(u_k^B, \bar{u}^B)}{h(u_k^B, \bar{u}^C)h(\bar{u}^B, u_k^B)}\Big] \\
\mathcal{M}^{(v)}(v_j^B, v_k^C) &= h(v_k^C, \bar{u}^B)h(\bar{v}^B, v_k^C)\Big[t(v_j^B, v_k^C) \\
&\quad + \kappa t(v_k^C, v_j^B)\frac{f(v_k^C, \bar{u}^C)}{f(v_k^C, \bar{u}^B)}\frac{h(\bar{v}^C, v_k^C)h(v_k^C, \bar{v}^B)}{h(v_k^C, \bar{v}^C)h(\bar{v}^B, v_k^C)}\Big]
\end{aligned}
$$

$a \times a$ block (first equation); $b \times b$ block (second equation)

Off-diagonal blocks

$$\mathcal{M}^{(u)}(u_j^C, v_k^C) = \kappa t(v_k^C, u_j^C)h(v_k^C, \bar{u}^C)h(\bar{v}^C, v_k^C) \qquad a \times b \text{ block}$$

$$\mathcal{M}^{(v)}(v_j^B, u_k^B) = t(v_j^B, u_k^B)h(\bar{v}^B, u_k^B)h(u_k^B, \bar{u}^B) \qquad b \times a \text{ block}$$