Spin chains based on (quantum) gl(3) algebras: Bethe vectors, scalar products and form factors

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Spin chains based on (quantum) \( gl(3) \) algebras: Bethe vectors, scalar products and form factors

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General background

Plan of the talk
Notations
Bethe vectors
Correlation functions
Multiple actions of \( T_{ij}(\vec{x}) \) on BVs
Scalar products of BVs
Form factors (off-diagonal case)
Form factors (diagonal case)
Form factors (remaining)
Summary
Case of trigonometric \( R \)-matrices
Conclusion

General background: Integrable spin chains

Rational or trigonometric \( 9 \times 9 \) \( R \)-matrix

\[ R(x, y) \in V \otimes V \text{ with } V = \text{End}(\mathbb{C}^3) \]

\( R(x, y) \) obeys \textbf{Yang-Baxter equation} (in \( V \otimes V \otimes V \))

\[ R^{12}(x_1, x_2) R^{13}(x_1, x_3) R^{23}(x_2, x_3) = R^{23}(x_2, x_3) R^{13}(x_1, x_3) R^{12}(x_1, x_2) \]

It is associated to a quantum group \( \mathcal{A} \) which is:

- The Yangian \( \mathcal{A} = Y(gl_3) \) when \( R(x, y) \) is rational (XXX chain)
- The affine quantum group \( \mathcal{A} = U_q(\widehat{gl}_3) \) when \( R(x, y) \) is trigonometric (XXZ chain)
General background: Integrable spin chains

**Rational or trigonometric 9 × 9 R-matrix**

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**For the talk, rational R-matrix:**

\[ R(x, y) = I + g(x, y)P \in \text{End}(\mathbb{C}^3) \otimes \text{End}(\mathbb{C}^3) \text{ and } g(x, y) = \frac{c}{x - y} \]

\( I \) is the identity matrix, \( P \) is the permutation matrix between two spaces \( \text{End}(\mathbb{C}^3) \), \( c \) is a constant.
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**Monodromy matrix**

$$T(x) = \sum_{i,j=1}^{3} e_{ij} \otimes T_{ij}(x) \in \text{End}(\mathbb{C}^3) \otimes \mathcal{A}$$

$T(x)$ obeys the RTT commutation relations:

$$R^{12}(x, y) \ T^1(x) \ T^2(y) = T^2(y) \ T^1(x) \ R^{12}(x, y)$$

This defines the quantum group $\mathcal{A}$

It leads to an integrable model through the transfer matrix

$$t(x) = tr_0 \ T^0(x) = T_{11}(x) + T_{22}(x) + T_{33}(x) \in \mathcal{A}$$

$$[t(x), t(y)] = 0$$

$$\text{Monodromy matrix}$$
Choice of a $Y(gl_3)$ (lowest weight) representation:

$$T_{jj}(w)|0\rangle = \lambda_j(w)|0\rangle, \quad j = 1, 2, 3 \quad T_{ij}(w)|0\rangle = 0, \quad 1 \leq j < i \leq 3$$

Up to normalisation $T(w) \rightarrow \lambda_2^{-1}(w)T(w)$, only need the ratios

$$r_1(w) = \frac{\lambda_1(w)}{\lambda_2(w)}, \quad r_3(w) = \frac{\lambda_3(w)}{\lambda_2(w)}.$$

where $r_1$ and $r_3$ are free functional parameters.
Aim

Compute the correlation functions $< O_1 \cdot \cdot \cdot O_n > = tr(O_1 \cdot \cdot \cdot O_n)$ for some local operators $O_1, \cdot \cdot \cdot, O_n$

If one has a basis of the space of states $\mathcal{H}$, $\{|\psi>\}$, then it is enough to compute $< \psi'|O_1 \cdot \cdot \cdot O_n|\psi>$
Since we have a basis $O|\psi> = \sum <\psi'|O|\psi> |\psi'>$, and we need "only" $<\psi|\psi'>$
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Aim

- Compute Bethe vectors (BVs), eigenvectors of $t(x)$:

  $$ t(x) B^{a,b}(\bar{u}, \bar{v}) = \tau(x|\bar{u}, \bar{v}) B^{a,b}(\bar{u}, \bar{v}) \Rightarrow \text{Bethe ansatz eqs (BAE)} $$

- Action of $T_{ij}(\bar{x})$ on $B^{a,b}(\bar{u}, \bar{v})$

- Scalar product of off-shell BVs (without BAE)

- Form factors $C^{a,b}(\bar{t}, \bar{s}) T_{ij}(\bar{x}) B^{a,b}(\bar{u}, \bar{v})$
Plan of the talk

- Bethe vectors (BVs)
- Multiple actions of Yangian generators on BVs
- Scalar products of BVs
- Form factors and correlation functions
- Conclusion

Calculations are rather technical ⇒ results only!

Presentation for $Y(gl_3)$ but most of the results are valid for $U_q(gl_3)$ (see at the end)
Notations

Apart from the functions \( g(x, y) = \frac{c}{x - y} \), \( r_1(x) \) and \( r_3(x) \) we introduce

\[
\begin{align*}
  f(x, y) &= \frac{x - y + c}{x - y} , \\
  h(x, y) &= \frac{f(x, y)}{g(x, y)} , \\
  t(x, y) &= \frac{g(x, y)}{h(x, y)} .
\end{align*}
\]
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\]

- "bar" always denote sets of variables: \( \bar{w}, \bar{u}, \bar{v} \) etc..
- \(|.|\) is the dimension of a set: \( \bar{w} = \{w_1, w_2\} \Rightarrow |\bar{w}| = 2 \), etc...
- Individual elements of the sets have latin subscripts: \( w_j, u_k, \) etc..
- Subsets of variables are denoted by roman indices: \( \bar{u}_I, \bar{v}_I, \bar{w}_I, \) etc.
- Special case: \( \bar{u}_j = \bar{u} \setminus \{u_j\} \), \( \bar{w}_k = \bar{w} \setminus \{w_k\} \), etc...
Bethe vectors

Framework: Algebraic-Nested Bethe ansatz (Leningrad school 80’s)
[Faddeev, Kulish, Reshetikhin, Sklyanin, Takhtajan]

On-shell Bethe vectors

\[ t(x) \mathcal{B}^{a,b}(\bar{u}; \bar{v}) = \tau(x|\bar{u}; \bar{v}) \mathcal{B}^{a,b}(\bar{u}; \bar{v}) \]

\( \bar{u} = \{u_1, ..., u_a\} \) and \( \bar{v} = \{v_1, ..., v_b\} \) are the Bethe parameters. \( t(x) \)-eigenvectors provided \( \bar{u} \) and \( \bar{v} \) obey the Bethe equations (BAEs):

\[
\begin{align*}
    r_1(\bar{u}_I) &= \frac{f(\bar{u}_I, \bar{u}_\Pi)}{f(\bar{u}_\Pi, \bar{u}_I)} f(\bar{v}, \bar{u}_I), \\
    r_3(\bar{v}_I) &= \frac{f(\bar{v}_\Pi, \bar{v}_I)}{f(\bar{v}_I, \bar{v}_\Pi)} f(\bar{v}_I, \bar{u}).
\end{align*}
\]

that hold for arbitrary partitions of the sets \( \bar{u} \) and \( \bar{v} \) into subsets \( \{\bar{u}_I, \bar{u}_\Pi\} \) and \( \{\bar{v}_I, \bar{v}_\Pi\} \).
Known formulas: Trace formula ['07 Tarasov & Varchenko]

$$B^{a,b}(\bar{u}; \bar{v}) = tr \left( T(\bar{u}; \bar{v}) R(\bar{u}; \bar{v}) e_{21}^a \otimes e_{32}^b \right) \in \mathcal{Y}(gl_3)$$

where $T$ is some product of $T(x)$’s and $R$ of $R$-matrices.

Recursion formulas

$$\lambda_2(u_k) f(\bar{v}, u_k) B^{a+1,b}(\bar{u}; \bar{v}) = T_{12}(u_k) B^{a,b}(\bar{u}_k; \bar{v}) +$$

$$+ \sum_{i=1}^{b} g(v_i, u_k) f(\bar{v}_i, v_i) T_{13}(u_k) B^{a,b-1}(\bar{u}_k; \bar{v}_i),$$

$$\lambda_2(v_k) f(\bar{v}, v_k) B^{a,b+1}(\bar{u}; \bar{v}) = T_{23}(v_k) B^{a,b}(\bar{u}; \bar{v}_k) +$$

$$+ \sum_{j=1}^{a} g(v_k, u_j) f(u_j, \bar{u}_j) T_{13}(v_k) B^{a-1,b}(\bar{u}_j; \bar{v}_k).$$
Explicit formulas

$$B^{a,b}(\vec{u}; \vec{v}) = \sum \frac{K_k(\vec{v}_I|\vec{u}_I)}{\lambda_2(\vec{v}_\Pi)\lambda_2(\vec{u})} \frac{f(\vec{v}_\Pi, \vec{v}_I)f(\vec{u}_\Pi, \vec{u}_I)}{f(\vec{v}_\Pi, \vec{v})f(\vec{v}_I, \vec{u}_I)} T_{12}(\vec{u}_\Pi) T_{13}(\vec{u}_I) T_{23}(\vec{v}_\Pi)|0\rangle$$

Plus others with different order of $T_{12}$, $T_{13}$, $T_{23}$

The sums are taken over partitions of the sets $\vec{u} \Rightarrow \{\vec{u}_I, \vec{u}_\Pi\}$ and $\vec{v} \Rightarrow \{\vec{v}_I, \vec{v}_\Pi\}$ with $0 \leq |\vec{u}_I| = |\vec{v}_I| = k \leq \min(a, b)$.

$K_k(\vec{v}_I|\vec{u}_I)$ is the Izergin–Korepin determinant

$$K_k(\vec{x}|\vec{y}) = \prod_{\ell < m} g(x_\ell, x_m)g(y_m, y_\ell) \cdot h(\vec{x}, \vec{y}) \det_k [t(x_i, y_j)].$$
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**Scalar products of BVs**

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**All these formulas are related**

- Explicit expressions obey the recursion formulas
- Trace formula obeys the recursion formulas
- Recursion formulas uniquely fix the BVs, once \( \mathcal{B}_{a,0}(\vec{u},.) \) or \( \mathcal{B}^0,b(.,\vec{v}) \) are known.

| Bethe vectors \( \mathcal{B}_{a,b}(\vec{u};\vec{v}) \), \(|\vec{u}| = a, |\vec{v}| = b\) |
|-----------------------------------|
| **On-shell BVs**: \( \vec{u}, \vec{v} \) obey BAES so that |
| \[ t(x) \mathcal{B}_{a,b}(\vec{u};\vec{v}) = \tau(x|\vec{u};\vec{v}) \mathcal{B}_{a,b}(\vec{u};\vec{v}) \] |
| **Off-shell BVs**: \( \vec{u}, \vec{v} \) are left free |

| Dual Bethe vectors \( \mathcal{C}_{a,b}(\vec{u};\vec{v}) \), \(|\vec{u}| = a, |\vec{v}| = b\) |
|-----------------------------------|
| **On-shell dual BVs**: \( \vec{u}, \vec{v} \) obey BAES so that |
| \[ \mathcal{C}_{a,b}(\vec{u};\vec{v}) t(x) = \tau(x|\vec{u};\vec{v}) \mathcal{C}_{a,b}(\vec{u};\vec{v}) \] |
| **Off-shell dual BVs**: \( \vec{u}, \vec{v} \) are left free |
Correlation functions

How to compute $\mathcal{O}_{\mathcal{C}, \mathcal{B}} = \langle \mathcal{C} | \mathcal{O} | \mathcal{B} \rangle$?

If $|\mathcal{B}\rangle$ is a complete basis (of transfer matrix eigenvectors), then

$$\mathcal{O} |\mathcal{B}\rangle = \sum_{\mathcal{B}'} \mathcal{O}_{\mathcal{B}\mathcal{B}'} |\mathcal{B}'\rangle$$  \hspace{1cm} (1)

→ what is needed is $\langle \mathcal{C} | \mathcal{B}' \rangle$ and (1)

Local operators: $\mathcal{O} = \sum_{\ell=1}^{L} \sum_{i,j=1}^{3} \mathcal{O}_{ij}^{(\ell)} e_{ij}^{\ell} \Rightarrow \langle \mathcal{C} | e_{ij}^{\ell} | \mathcal{B} \rangle$

Further simplification: QISM

Expression of $e_{ij}^{\ell}$, $i, j = 1, 2, 3$ and $\ell = 1, \ldots L$, in terms of monodromy entries $T_{kl}(x)$ [00 Maillet & Terras]:

$$e_{ij}^{\ell} = (t(0))^{\ell-1} T_{ij}(0) (t(0))^{-\ell}$$

⇒ we need "only" $T_{kl}(x) \mathcal{B}^{a,b}(\bar{u}; \bar{v})$ and $\mathcal{C}^{a,b}(\bar{w}; \bar{z}) \mathcal{B}^{a,b}(\bar{u}; \bar{v})$
Multiple actions of $T_{ij}(\bar{x})$ on $\mathbb{B}^{a,b}(\bar{u}; \bar{v})$

$|\bar{x}| = n, \quad \{\bar{u}, \bar{x}\} = \bar{\eta}, \quad |\bar{\eta}| = a + n; \quad \{\bar{v}, \bar{x}\} = \bar{\xi}, \quad |\bar{\xi}| = b + n$

$$T_{13}(\bar{x})\mathbb{B}^{a,b}(\bar{u}; \bar{v}) = \lambda_2(\bar{x}) \mathbb{B}^{a+n,b+n}(\bar{\eta}; \bar{\xi}).$$

$$T_{12}(\bar{x})\mathbb{B}^{a,b}(\bar{u}; \bar{v}) = (-1)^n \lambda_2(\bar{x}) \sum f(\bar{\xi}_I, \bar{\xi}_I) K_n(\bar{\xi}_I|\bar{x} + c) \mathbb{B}^{a+n,b+n}(\bar{\eta}; \bar{\xi}_I).$$

Sum on partitions $\bar{\xi} = \{\bar{\xi}_I; \bar{\xi}_II\}$ with $|\bar{\xi}_I| = n$

$$T_{23}(\bar{x})\mathbb{B}^{a,b}(\bar{u}; \bar{v}) = (-1)^n \lambda_2(\bar{x}) \sum f(\bar{\eta}_I, \bar{\eta}_II) K_n(\bar{x}|\bar{\eta}_I + c) \mathbb{B}^{a,b+n}(\bar{\eta}_II; \bar{\xi}).$$

Sum on partitions $\bar{\eta} = \{\bar{\eta}_I; \bar{\eta}_II\}$ with $|\bar{\eta}_I| = n$

Imply recursion relations as a subcase (n=1)
Similar expressions for any $T_{ij}(\bar{x})$ and for dual BVs
Scalar products of BVs

\[ S_{a,b} \equiv S_{a,b}(\bar{u}^C, \bar{u}^B|\bar{v}^C, \bar{v}^B) = \mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B) \]

Superscripts \(B\) and \(C\) to denote different sets of parameters!

General formula given by Reshetikhin

\[
S_{a,b} = \sum r_1(\bar{u}^B_I)r_1(\bar{u}^C_{\Pi})r_3(\bar{v}^B_I)r_3(\bar{v}^C_{\Pi})
\times f(\bar{u}^C_I, \bar{u}^C_{\Pi})f(\bar{u}^B_{\Pi}, \bar{u}^B_I)f(\bar{v}^C_I, \bar{v}^C_{\Pi})f(\bar{v}^B_I, \bar{v}^B_{\Pi})f(\bar{v}^C_I, \bar{u}^C_I)f(\bar{v}^B_{\Pi}, \bar{u}^B_I)
\times Z_{a-k,n}(\bar{u}^C_{\Pi}; \bar{u}^B_I|\bar{v}^C_I; \bar{v}^B_I)Z_{k,b-n}(\bar{u}^B_I; \bar{u}^C_I|\bar{v}^B_{\Pi}; \bar{v}^C_{\Pi})
\]

\(\bar{u}^B = \{\bar{u}^B_I, \bar{u}^B_{\Pi}\}\), \(\bar{u}^C = \{\bar{u}^C_I, \bar{u}^C_{\Pi}\}\) with \(|\bar{u}^B_I| = |\bar{u}^C_I| = k\) for \(k = 0, \ldots, a\)

\(\bar{v}^B = \{\bar{v}^B_I, \bar{v}^B_{\Pi}\}\), \(\bar{v}^C = \{\bar{v}^C_I, \bar{v}^C_{\Pi}\}\) with \(|\bar{v}^B_I| = |\bar{v}^C_I| = n\) for \(n = 0, \ldots, b\).

\(Z_{a,b}\) so-called highest coefficient

\[
Z_{a,b}(\bar{t}; \bar{x}|\bar{s}; \bar{y}) = (-1)^b \sum K_b(\bar{s} - c|\bar{w}_I)K_a(\bar{w}_{\Pi}|\bar{t})K_b(\bar{y}|\bar{w}_I)f(\bar{w}_I, \bar{w}_{\Pi}).
\]

But \(S_{a,b}\) difficult to handle.....
Here we consider the scalar product of an on-shell Bethe vector
\[ t(x) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B) = \tau(x|\bar{u}^B, \bar{v}^B) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B) \] and BAEs with a twisted dual on-shell Bethe vector
\[ C^{a,b}_\kappa(\bar{u}^C; \bar{v}^C) t_\kappa(x) = \tau_\kappa(x|\bar{u}^C, \bar{v}^C) C^{a,b}_\kappa(\bar{u}^C; \bar{v}^C) \]
with twisted BAEs
\[ t_\kappa(x) = \text{tr}(M T(x)) \quad \text{with} \quad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & 1 \end{pmatrix} \]
\[ t_\kappa(x) = T_{11}(x) + \kappa T_{22}(x) + T_{33}(x) \]
\[ S_{a,b} \equiv S_{a,b}(\bar{u}^C, \bar{u}^B|\bar{v}^C, \bar{v}^B) = C^{a,b}_\kappa(\bar{u}^C; \bar{v}^C) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B) \]
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\[
S_{a,b} = f(\vec{v}^C, \vec{u}^C)f(\vec{v}^B, \vec{u}^B)t(\vec{v}^C, \vec{u}^B) \Delta'_a(\vec{u}^C) \Delta_a(\vec{u}^B) \Delta'_b(\vec{v}^C) \Delta_b(\vec{v}^B) \times \det_{a+b} \mathcal{M},
\]

\[
\Delta'_n(\vec{x}) = \prod_{j>k} g(x_j, x_k), \quad \Delta_n(\vec{y}) = \prod_{j<k} g(y_j, y_k).
\]

\( \mathcal{M} \) is a \((a + b) \times (a + b)\) matrix. For \( \vec{y} = \{ \vec{u}^B, \vec{v}^C \} \):

\[
\mathcal{M}_{j,k} = \frac{c}{g(y_k, \vec{u}^C)g(\vec{v}^C, y_k)} \frac{\partial \tau_{\kappa}(y_k | \vec{u}^C, \vec{v}^C)}{\partial u^C_j}, \quad j = 1, \ldots, a,
\]

\[
\mathcal{M}_{a+j,k} = \frac{-c}{g(y_k, \vec{u}^B)g(\vec{v}^B, y_k)} \frac{\partial \tau(y_k | \vec{u}^B, \vec{v}^B)}{\partial v^B_j}, \quad j = 1, \ldots, b.
\]

Similar expression for \( S_{a,b} \) when considering a general twist

\[
t_{\tilde{\kappa}}(x) = \kappa_1 T_{11}(x) + \kappa_2 T_{22}(x) + \kappa_3 T_{33}(x)
\]

but up to terms \((\kappa_i - 1)(\kappa_j - 1), i, j = 1, 2, 3\)
Form factors (off-diagonal case)

\[ F_{a,b}^{(i,j)}(z) = C^{a',b'}(\bar{u}^C; \bar{v}^C) T_{ij}(z) B^{a,b}(\bar{u}^B; \bar{v}^B), \]

\[ a' = a + \delta_{i1} - \delta_{j1}, \quad b' = b + \delta_{j3} - \delta_{i3}, \quad i,j = 1,2,3. \]

Both \( C^{a',b'}(\bar{u}^C; \bar{v}^C) \) and \( B^{a,b}(\bar{u}^B; \bar{v}^B) \) are on-shell Bethe vectors.

\[ F_{a,b}^{(1,2)}(z) = \mathcal{H}_{a',b} \det \mathcal{N}, \]

\[ \mathcal{H}_{a',b} = \frac{\Delta_{a'}(\bar{u}^C) \Delta_{b}(\bar{v}^B) \Delta_{a+b+1}(\bar{x})}{h(\bar{v}^C, \bar{u}^B)}, \quad \bar{x} = \{ \bar{u}^B, \bar{v}^C, z \} \]

\[ \mathcal{N}_{j,k} = \frac{c}{g(x_k, \bar{u}^C)g(\bar{v}^C, x_k)} \frac{\partial \tau(x_k | \bar{u}^C, \bar{v}^C)}{\partial u^C_j}, \quad j = 1, \ldots, a', \]

\[ \mathcal{N}_{a'+j,k} = \frac{-c}{g(x_k, \bar{u}^B)g(\bar{v}^B, x_k)} \frac{\partial \tau(x_k | \bar{u}^B, \bar{v}^B)}{\partial v^B_j}, \quad j = 1, \ldots, b. \]

Similar expression for all \( F_{a,b}^{(i,j)}(z), |i-j| = 1. \)
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Form factors (diagonal case)

$$J_{a,b}^{(s,s)}(z) = (-1)^b \mathcal{H}_{a',b} \cdot \det_{a+b+1} \mathcal{N}^{(s)},$$

$$\mathcal{N}_{j,k}^{(s)} = \mathcal{N}_{j,k}, \quad j = 1, \ldots, a + b, \quad k = 1, \ldots, a + b + 1;$$

The last line of $\mathcal{N}^{(s)}$ depends on $s$, for instance:

$$\mathcal{N}_{a+b+1,k}^{(1)} = h(x_k, \vec{u}^B) h(\vec{v}^C, x_k) \left\{ \frac{u_k^B}{c} \left( \frac{f(\vec{v}^B, u_k^B)}{f(\vec{v}^C, u_k^B)} - 1 \right) - 1 \right\},$$

$$k = 1, \ldots, a;$$

$$\mathcal{N}_{a+b+1,a+k}^{(1)} = h(x_{a+k}, \vec{u}^B) h(\vec{v}^C, x_{a+k}) \left\{ \frac{v_k^C + c}{c} \left( \frac{f(v_k^C, \vec{u}^C)}{f(v_k^C, \vec{u}^B)} - 1 \right) - 1 \right\},$$

$$k = 1, \ldots, b;$$

$$\mathcal{N}_{a+b+1,a+b+1}^{(1)} = \frac{r_1(z) f(\vec{u}^B, z)}{g(\vec{v}^C, z) g(z, \vec{u}^B)}$$

$\mathcal{N}^{(3)}$ has a similar expression;

$$\mathcal{N}_{a+b+1,k}^{(2)} = h(x_k, \vec{u}^B) h(\vec{v}^C, x_k), \quad k = 1, \ldots, a + b + 1.$$
Form factors (remaining)

- $F^{(i,j)}_{a,b}(z), |i - j| = 1$, are computed by brute force, using multiple action of $T_{ij}$'s

- $F^{(s,s)}_{a,b}(z)$ are computed using the trick of "twisted BVs":

$$F^{(s,s)}_{a,b}(z) = \frac{d}{d\kappa_s} \left[ \mathbb{C}^{a,b}_\kappa (\bar{u}^C; \bar{V}^C)(t_\kappa(z) - t(z)) \mathbb{B}^{a,b}(\bar{u}^B; \bar{V}^B) \right]_{\kappa = 1}$$

- $F^{(1,3)}_{a,b}(z)$ and $F^{(3,1)}_{a,b}(z)$ are not computable by these two methods: one needs a new one. We may have found a general one that could help to compute all the form factors in an easy way. More information soon...
Summary

For models with $GL(3)$ invariant $R$-matrix, we got:

- Explicit expressions for (off-shell) Bethe vectors and their duals
- Multiple action of monodromy elements on these BVs
  
  Both results in term of Izergin-Korepin determinants and sums of partitions of sets of Bethe parameters

- Calculation of the scalar product of (twisted) on-shell BVs
- Calculation of the form factors of $T_{ij}(x)$, $i, j = 1, 2, 3$
  
  Both results in term of a single determinant (and product of scalar functions)
Case of trigonometric $R$-matrices

- Explicit expression of BVs for $U_q(gl_N)$ arXiv:1310.3253
  Use of projectors method in current realization of $U_q(gl_N)$, see works of Khoroshkin, Pakuliak and collaborators

- Multiple actions of $T_{ij}(\vec{w})$ for $U_q(gl_3)$ arXiv:1304.7602

Conclusion: still a lot to do...

- Calculation of the scalar product of generic off-shell BVs (as a single determinant)
  See for instance recent work of Wheeler, arXiv:1306.0552
- Complete calculation of correlation functions, asymptotics, etc...
- Generalization to other models
  - Calculation of the form factors of $T_{jk}(x)$, for $U_q(gl_3)$
  - Case of $U_q(gl_N)$ algebras
Thank you!
Spin chains based on (quantum) \textit{gl}(3) algebras: Bethe vectors, scalar products and form factors

Eric Ragoucy

General background

Plan of the talk

Notations

Bethe vectors

Correlation functions

Multiple actions of $T_{ij} (\vec{x})$ on BVs

Scalar products of BVs

Form factors (off-diagonal case)

Form factors (diagonal case)

Form factors (remaining)

Summary

Case of trigonometric $\textit{R}$-matrices

Conclusion

\[ \mathcal{F}_{a,b}^{(1,3)} (z) \text{ form factor} \]

\[ \mathcal{F}_{a,b}^{(1,3)} (z) = C^{a+1,b+1} (\vec{u}^C ; \vec{v}^C) T_{13} (z) \mathbb{B}^{a,b} (\vec{u}^B ; \vec{v}^B) \]

\[ = (-1)^{b+1} \mathcal{H}_{a+1,b} \cdot \det \mathcal{N}_{a+b+2}^{(1,3)}, \]

\[ \mathcal{N}_{j,k}^{(1,3)} = \mathcal{N}_{j,k}, \quad j, k = 1, \ldots, a + b + 1; \]

\[ \mathcal{N}_{a+b+2,k}^{(1,3)} = (-1)^{b+1} r_3 (x_k) \frac{h (x_k, \vec{v}^B)}{g (x_k, \vec{u}^B)} + h (\vec{v}^B, x_k) h (x_k, \vec{u}^B). \]
Explicit formulas

$$B^{a,b}(\vec{u}; \vec{v}) = \sum \frac{K_k(\vec{v}_I | \vec{u}_I)}{\lambda_2(\vec{v}_I) \lambda_2(\vec{u})} \frac{f(\vec{v}_I, \vec{v}_I) f(\vec{u}_I, \vec{u}_I)}{f(\vec{v}_I, \vec{u}_I) f(\vec{v}_I, \vec{u}_I)} T_{12}(\vec{u}_I) T_{13}(\vec{u}_I) T_{23}(\vec{v}_I) |0\rangle$$

$$B^{a,b}(\vec{u}; \vec{v}) = \sum \frac{K_k(\vec{v}_I | \vec{u}_I)}{\lambda_2(\vec{v}_I) \lambda_2(\vec{v})} \frac{f(\vec{v}_I, \vec{v}_I) f(\vec{u}_I, \vec{u}_I)}{f(\vec{v}_I, \vec{u}_I) f(\vec{v}_I, \vec{u}_I)} T_{23}(\vec{v}_I) T_{13}(\vec{v}_I) T_{12}(\vec{u}_I) |0\rangle$$

$$B^{a,b}(\vec{u}; \vec{v}) = \sum \frac{K_k(\vec{v}_I | \vec{u}_I)}{\lambda_2(\vec{v}_I) \lambda_2(\vec{v})} \frac{f(\vec{v}_I, \vec{v}_I) f(\vec{u}_I, \vec{u}_I)}{f(\vec{v}, \vec{u})} T_{13}(\vec{u}_I) T_{12}(\vec{u}_I) T_{23}(\vec{v}_I) |0\rangle$$

$$B^{a,b}(\vec{u}; \vec{v}) = \sum \frac{K_k(\vec{v}_I | \vec{u}_I)}{\lambda_2(\vec{v}_I) \lambda_2(\vec{v})} \frac{f(\vec{v}_I, \vec{v}_I) f(\vec{u}_I, \vec{u}_I)}{f(\vec{v}, \vec{u})} T_{13}(\vec{v}_I) T_{23}(\vec{v}_I) T_{12}(\vec{u}_I) |0\rangle$$

The sums are taken over partitions of the sets
$$\vec{u} \Rightarrow \{\vec{u}_I, \vec{u}_I\} \text{ and } \vec{v} \Rightarrow \{\vec{v}_I, \vec{v}_I\} \text{ with } 0 \leq |\vec{u}_I| = |\vec{v}_I| = k \leq \min(a, b).$$

$$K_k(\vec{v}_I | \vec{u}_I)$$ is the Izergin–Korepin determinant

$$K_k(\vec{x} | \vec{y}) = \prod_{\ell < m}^{k} g(x_\ell, x_m) g(y_m, y_\ell) \cdot h(\vec{x}, \vec{y}) \det_k [t(x_i, y_j)].$$
Spin chains based on (quantum) \( gl(3) \) algebras: Bethe vectors, scalar products and form factors

Eric Ragoucy

**General background**

**Plan of the talk**

**Notations**

**Bethe vectors**

**Correlation functions**

**Multiple actions of** \( T_{ij}(\bar{x}) \) **on BVs**

**Scalar products of BVs**

**Form factors** **(off-diagonal case)**

**Form factors** **(diagonal case)**

**Form factors** **(remaining)**

**Summary**

**Case of trigonometric** \( R \)-matrices

**Conclusion**

The matrix \( \mathcal{M} \)

### Diagonal blocks

\[
\mathcal{M}^{(u)}(u^C_j, u^B_k) = h(\bar{v}^C, u^B_k)h(u^B_k, \bar{u}^C) \left[ \kappa t(u^B_k, u^C_j) + t(u^C_j, u^B_k) \frac{f(\bar{v}^B, u^B_k)h(\bar{u}^C, u^B_k)h(u^B_k, \bar{u}^B)}{f(\bar{v}^C, u^B_k)h(u^B_k, \bar{u}^B)} \right]
\]

**a \times a block**

\[
\mathcal{M}^{(v)}(v^B_j, v^C_k) = h(v^C_k, \bar{u}^B)h(\bar{v}^B, v^C_k) \left[ t(v^B_j, v^C_k) + \kappa t(v^C_k, v^B_j) \frac{f(v^C_k, \bar{u}^B)h(v^C_k, \bar{v}^B)h(\bar{v}^B, v^C_k)}{f(v^C_k, \bar{u}^B)h(v^C_k, \bar{v}^B)h(\bar{v}^B, v^C_k)} \right]
\]

**b \times b block**

### Off-diagonal blocks

\[
\mathcal{M}^{(u)}(u^C_j, v^C_k) = \kappa t(v^C_k, u^C_j)h(\bar{v}^C, \bar{u}^C)h(v^C_k, \bar{u}^C) \quad a \times b \text{ block}
\]

\[
\mathcal{M}^{(v)}(v^B_j, u^B_k) = t(v^B_j, u^B_k)h(\bar{v}^B, u^B_k)h(u^B_k, \bar{u}^B) \quad b \times a \text{ block}
\]