Free fields approach to form factors of descendants

Yaroslav Pugai

Landau Institute

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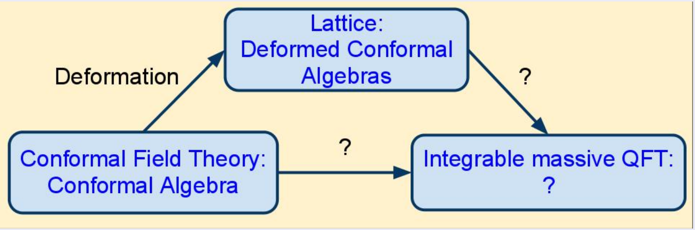
Dynamical Symmetry in 2D integrable models

Space of fields



Number of local operators

- CFT: Character of Virasoro algebra $Tr\ q^{L_0}$
- Lattice: Trace of corner transfer matrix $Tr\ q^{L_0}$
- Massive FT: $\sum q^{\#spin-s\ solutions\ form\ factor\ equations}$

Massive Integrable 2D QFT

Perturbed two dimensional CFT ($c = 1 - \frac{6}{\xi(\xi+1)}$, $\Delta_\Phi = \frac{\xi-1}{(\xi+1)}$)

$$\mathcal{A} = \mathcal{A}_{CFT} + \lambda \int d^2y \ \Phi(y)$$

$$\langle O_1(x) O_2(0) \rangle_{massive} = \langle O_1(x) O_2(0) e^{\lambda \int d^2y \ \Phi(y)} \rangle_{conformal}$$

S-matrix description

$$S(\beta) = \frac{\sinh \beta + i \sin \pi\xi}{\sinh \beta - i \sin \pi\xi}$$

$$\langle O_1(x) O_2(0) \rangle_{massive} = \sum \int d\beta \cdots \langle O_1(x) | \beta, \ldots \rangle \langle \beta, \ldots | O_2(0) \rangle$$

Exact form factors

Axioms ($E = m\cosh\ \beta, \quad P = m\sinh\ \beta$)

$$
\begin{aligned}
&\langle O|\beta_1, \beta_2, \ldots\rangle = S(\beta_1 - \beta_2)\langle O|\beta_2, \beta_1, \ldots\rangle \\
&\langle O|\beta_1 + A, \beta_2 + A, \ldots\rangle = e^{sA}\langle O|\beta_1, \beta_2, \ldots\rangle \\
&\langle O|\beta_1 + 2\pi i, \beta_2, \ldots, \beta_n\rangle = \langle O|\beta_2, \ldots, \beta_n, \beta_1\rangle \\
&Res_{\beta_1=\beta_2+i\pi}\langle O|\beta_1, \beta_2, \beta_3 \ldots\rangle = i(1 - \prod S(\beta_2 - \beta_j))\langle O|\beta_3, \ldots\rangle \\
&\ldots
\end{aligned}
$$

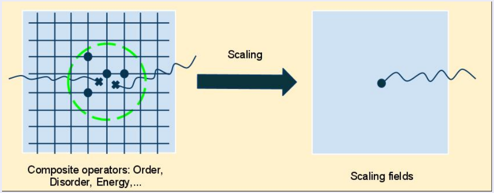
Exact answers:

- Primaries are studied well
- Descendants: there are questions

Composite operators

Composite operators

$\sigma(x_1)\cdots\mu(x_n)\cdots = \sum O_j(x_0), \quad |x_m - x_n| << R_c$



Number of fields is "the same" as for $\lambda = 0$, but ...

$$\langle O_1(x)O_2(0)\rangle_{massive} = \langle O_1(x)O_2(0)e^{\lambda\int d^2y\ \Phi(y)}\rangle_{conformal}$$

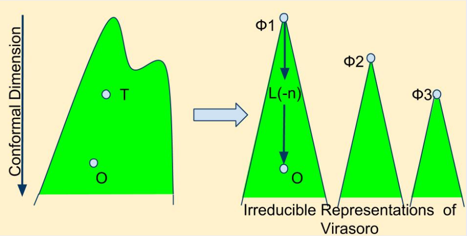
- No Virasoro algebra
- No $Vir \times \overline{Vir}$: $\bar{\partial}T \neq 0 \quad (\sim \lambda\partial\Phi)$
- Sectors are mixed $[O] \sim [O] \oplus [O'] \ [\Phi]$ (resonances)

Space of States in CFT (BPZ)

Virasoro Algebra $T(z) = \sum L_n z^{-n-2}$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^2-1)n\delta_{n+m}$$

Primaries Φ_k and Descendants $L_{-n}\cdots\bar{L}_{-m}\cdots\Phi_k$



$$L_0|\Phi_k\rangle = \Delta_k|\Phi_k\rangle$$

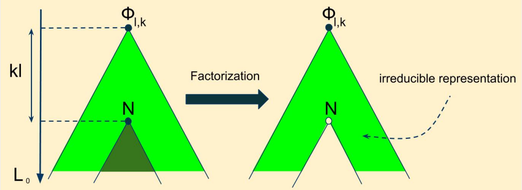
$$L_n|\Phi_k\rangle = 0 \quad (n > 0)$$

Null vectors in CFT

Kac spectrum Φ_{lk} $\left(c = 1 - \frac{6}{\xi(\xi+1)}\right)$

$$\Delta_{lk} = \frac{((\xi+1)l - \xi k)^2 - 1}{4\xi(\xi+1)}$$

Null vectors $D_{lk} = (L_{-1}^{lk} + \cdots)|\Phi_{lk}\rangle$



$$L_0|D_{lk}\rangle = (\Delta_{lk} + lk)\,|D_{lk}\rangle$$

$$L_n|D_{lk}\rangle = 0, \quad n > 0$$

Oscillator realization

Oscillators and zero modes $[a_m, a_n] = m\delta_{m+n}, \quad [\mathcal{Q}, a_0] = i$

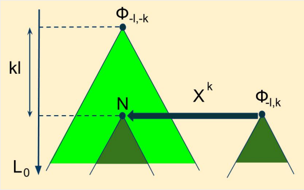
- Free field
$$\phi(z) = \mathcal{Q} - ia_0 \log z + \sum_{m\neq 0} \frac{a_m}{im} z^{-m}$$

- Screening currents
$$X = \oint \frac{dz}{2\pi i z} e^{-i\sqrt{2\frac{\xi+1}{\xi}}\phi(z)}$$

- Virasoro algebra: $[T(z), X] = 0$
$$T(z) = \frac{1}{2}\partial\phi(z)^2 + \frac{1}{\sqrt{2\xi(\xi+1)}}\partial^2\phi(z)$$

Null vectors in Fock space $\mathcal{F}_{l,k} = \{a_{-n_1} \cdots a_{-n_m} \Phi_{l,k}\}$

Singular vectors $[X^k, T(z)]\big|_{\mathcal{F}_{-l,k}} = 0$



- Level One $\langle\Phi_{11}|D_{11} = \langle\Phi_{11}|a_1$
- Level Two $\langle\Phi_{12}|D_{12} = \langle\Phi_{12}|(a_2 + \frac{\xi+1}{\xi} a_1^2)$
- General case $D_{l,k}$ - in terms of Jack polynomials $(a_n \to \sum_j x_j^n)$

DVA in CTM approach

Deformed Virasoro (CFT limit: $x \to 1, \quad z \sim$ fixed)

- Oscillators
$$[a_m, a_n] = \frac{1}{m}\frac{(x^{2\xi m}-x^{-2\xi m})(x^{2(\xi+1)m}-x^{-2(\xi+1)m})}{x^m+x^{-m}}\delta_{m+n}$$
- Screening currents $U(z) \sim e^{-\sum \frac{x^n+x^{-n}}{x^{\xi n}+x^{-\xi n}}a_n z^{-n}}$
$$X = \int \frac{dz}{2\pi i z}U(z)$$
- Deformed Virasoro Algebra: $[T, X] = 0$
$$T(z) = \Lambda(zx) + \Lambda^{-1}(zx^{-1})\,, \quad \Lambda(z) = e^{\sum a_n z^{-n}}$$
- Null vectors - Macdonald polynomials $q = x^{2\xi}, t = x^{2(\xi+1)}$

Form factors from off-critical lattice (XYZ, RSOS)

Scaling limit $x -> 1$, $z = x^{2i\beta}$

- Diagonalization of Hamiltonian $HT(x^{2i\beta}) = E(\beta)T(x^{2i\beta})$

$$E(\beta) \longrightarrow M \cosh \beta$$

- ZF algebra $T(x^{2i\beta_1})T(x^{2i\beta_2}) = S(\beta_{12})T(x^{2i\beta_2})T(x^{2i\beta_1})$

$$S(\beta) \longrightarrow \frac{\sinh \beta + i \sin \pi\xi}{\sinh \beta - i \sin \pi\xi}$$

- Form factors of primaries

$$Tr_{\mathcal{F}_{lk}} \left(x^{4L_0} T(x^{2i\beta_1}) \cdots T(x^{2i\beta_n}) \right) \longrightarrow \langle \Phi_{l,k} | \beta_1, \cdots, \beta_n \rangle$$

DVA at $x \to e^{-\frac{i\pi}{2}}$

Extra bosonic field $B = \exp \sum b_k z^{-k}$

$$\langle B(z)B(w)\rangle = \langle \Lambda(z)\Lambda(w)\rangle^{-1}$$
$$t(\beta) = T(x^{2i\beta})B(x^{2i\beta})$$

Form factors of primaries

$$\lim_{x\to -i} \langle \Phi_{l,k}|t(\beta_1)\cdots t(\beta_n)|\Phi_{l,k}\rangle =$$
$$= \lim_{x\to 1} Tr_{\mathcal{F}_{lk}} x^{4L_0} T(x^{2i\beta_1})\cdots T(x^{2i\beta_n}) \times Tr_{\mathcal{F}_{lk}} x^{4L_0} B(x^{2i\beta_1})\cdots B(x^{2i\beta_n})$$
$$= \langle \Phi_{l,k}|\beta_1, \ldots \beta_n\rangle / \prod_{i<j} R(\beta_i - \beta_j)$$

$R(\beta)$- two particle minimal form factor

Feigin-Lashkevich'09

Algebra $t(\beta) = e^{a_0}\lambda_-(\beta) + e^{-a_0}\lambda_+(\beta)$

$$\langle \lambda_+(\beta)\lambda_+(0)\rangle = \langle \lambda_-(\beta)\lambda_-(0)\rangle = 1$$
$$\langle \lambda_-(\beta)\lambda_+(0)\rangle = \langle \lambda_+(\beta)\lambda_-(0)\rangle = 1 + \frac{2i\sin\pi\xi}{e^{\beta} - e^{-\beta}}$$

Heisenberg Descendants a_n

- Action of oscillators $[\lambda_\pm(\beta), a_n] = (\mp)^{n+1} e^{n\beta}\lambda_\pm(\beta)$
- Left descendants $(n > 0)$
$$\langle a_{-n}\Phi_{l,k}|\beta_1, \cdots, \beta_n\rangle \rightarrow \langle \Phi_{l,k}|a_n t(\beta_1)\cdots t(\beta_n)|\Phi_{l,k}\rangle$$
- Right descendants $(n > 0)$
$$\langle \bar{a}_{-n}\Phi_{l,k}|\beta_1, \cdots, \beta_n\rangle \rightarrow \langle \Phi_{l,k}|t(\beta_1)\cdots t(\beta_n) a_{-n}|\Phi_{l,k}\rangle$$
- Left-right "interaction" $[a_{2n}, \bar{a}_{2m}] \sim \delta_{n,m}$

Example: level two null vectors

Level two descendant form factors: $D_{12} = a_1^2 + const \times a_2$

$$\langle\Phi_\alpha|a_2|\Phi_\alpha\rangle = 0\,,$$
$$\langle\Phi_\alpha|a_2 t(\beta_1)|\Phi_\alpha\rangle = 2ie^{2\beta_1}\sin\pi\alpha\,,$$
$$\langle\Phi_\alpha|a_2 t(\beta_1)t(\beta_2)|\Phi_\alpha\rangle = 2i\big((e^{2\beta_1}+e^{2\beta_2})\sin 2\pi\alpha + 2e^{\beta_1+\beta_2}\sin\pi\xi\big)$$

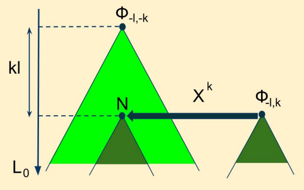
$$\langle\Phi_\alpha|a_1^2|\Phi_\alpha\rangle = 0\,,$$
$$\langle\Phi_\alpha|a_1^2 t(\beta_1)|\Phi_\alpha\rangle = 2e^{2\beta_1}\cos\pi\alpha\,,$$
$$\langle\Phi_\alpha|a_1^2 t(\beta_1)t(\beta_2)|\Phi_\alpha\rangle = 4(e^{\beta_1}+e^{\beta_2})^2\cos^2\pi\alpha\,,$$

For $\alpha = -(\xi+1)/2$ there is a null vector among Φ_{12} descendants:

$$\langle\Phi_{12}|D_{12}t(\beta_1)\cdots t(\beta_n)|\Phi_{12}\rangle = 0 \quad n = 0,1,2,\ldots$$

$$\langle\Phi_{12}|D_{12} = \langle\Phi_{12}|(a_1^2 + i\tan\frac{\pi\xi}{2}a_2)$$

Structure of the space of descendants



Questions

- Arbitrary level $k \times l$ null vectors $D_{lk} = a_1^{kl} + \cdots$?

$$\langle \Phi_{lk} | D_{lk} t(\beta_1) \ldots t(\beta_n) | \Phi_{lk} \rangle = 0$$

- Will D_{lk} create a null submodule?
- Currents conservation $\bar{\partial} T_{2k+2} = \partial \Theta_{2k}$
- Higher eq. of motion $\frac{d}{d\alpha} D_{lk} \bar{D}_{lk} \Phi_\alpha \Big|_{\alpha=\alpha_{lk}} = ?$

Macdonald Polynomials (q,t)

Definition

- $\lambda = \{\lambda_1, \lambda_2, \ldots\}, \quad \lambda_1 \geq \lambda_2 \geq \cdots, \sum \lambda_j = n$
- $T_{q,i} f(x_1, \ldots, x_i, \ldots, x_N) = f(x_1, \cdots, qx_i, \cdots, x_N)$
- $P_\lambda(x_1, \ldots, x_N)$ - symmetric polynomial of degree n

$$\sum_i \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} T_{q,i} P_\lambda = \sum t^{N-i} q^{\lambda_i} P_\lambda \,,$$

Examples $p_n = (1 - q^{-n}) \sum x_j^n \quad (\to a_n$ to get null vector for DVA)

$$P_{\{1\}} = p_1 \,, \quad P_{\{2,0\}} = p_1^2 + \frac{1+q}{1-q} p_2 \,, \quad P_{\{11\}} = p_1^2 - \frac{1+t}{1-t} p_2$$

General expression for null vectors

The proposal

Let $\lambda = \{k\}_{i=1}^{l}$ be $l \times k$ rectangular partition. The level $k \times l$ null vector descendant of Φ_{lk} is given by Macdonald polynomials $D_{lk} = P_\lambda\Big|_{p_n \to a_n}$ with

$$q = e^{-i\pi(\xi+1)}, \quad t = e^{-i\pi\xi}$$

For $n = 0, 1, 2, \ldots$:

$$\langle \Phi_{lk} | D_{lk} t(\beta_1) \cdots t(\beta_n) | \Phi_{lk} \rangle = 0$$

Integral representation

Intertwining operators

- Currents: $U(z) = U(z)^{(lattice)}|_{x=-i}$
- Charges $(x=-i,\ q=e^{-i\pi(\xi+1)})$

$$X^{(k)} = \oint \frac{dz_1}{2\pi i z_1} \cdots \oint \frac{dz_k}{2\pi i z_k} U(z_1)\cdots U(z_k) \times F(z_2/z_1)F(z_4/z_3)\cdots F(z_k/z_{k-1})$$

- $F(z) = \sum_{n=1}(-1)^{n-1}\frac{1-q^n}{1+q^n}z^n$
- $[X^{(k)}, t(z)] = 0$

Null Vectors in a single chirality

Statements on operators $X^{(k)}$

- Give integral representation for Macdonald polynomials with $q = e^{-i\pi(\xi+1)}, t = e^{-i\pi\xi}$
$$\langle \Phi_{l,k} | D_{l,k} = \langle \Phi_{l,-k} | \ X^{(k)}$$
- Commute with particle creation operators $t(\beta)$ and acts as
$$X^{(k)} \ t(\beta) \cdots |\Phi_{l,k}\rangle = 0$$
- Create a full tower of null vectors
$$a_n \ X^{(k)} \ t(\beta) \cdots |\Phi_{l,k}\rangle = 0$$

Two "chiralities" and resonances I

Currents: $U(z) = \sum \Psi_m z^{-m}$, $\Psi_m \Psi_n = -\Psi_{n+2}\Psi_{m-2}$

- $T_{2k+2} \longrightarrow \langle \Phi_{11}|\Psi_{2k+2} t(\beta) \cdots |\Phi_{11}\rangle$
- $\Theta_{2k} \longrightarrow \langle \Phi_{13}|\Psi_{2k} t(\beta) \cdots |\Phi_{13}\rangle$
- $\partial \longrightarrow a_1$
- $\bar{\partial} \longrightarrow \bar{a}_{-1}$

Conservation law $\bar{\partial} T_{2k+2} = \partial \Theta_{2k}$

$$\langle \Phi_{11}|\Psi_{2k+2} t(\beta) \cdots \bar{a}_{-1}|\Phi_{11}\rangle = \langle \Phi_{13}|\Psi_{2k} a_1 t(\beta) \cdots |\Phi_{13}\rangle$$

Resonance condition

$$\Delta_{\bar{\partial} T} = \Delta_{\partial \Theta} + (1 - \Delta_{\Phi})$$

Two "chiralities" and resonances I

Currents: $U(z) = \sum \Psi_m z^{-m}$, $\Psi_m \Psi_n = -\Psi_{n+2} \Psi_{m-2}$

- $T_{2k+2} \longrightarrow \langle \Phi_{11} | \Psi_{2k+2} t(\beta) \cdots | \Phi_{11} \rangle$
- $\Theta_{2k} \longrightarrow \langle \Phi_{13} | \Psi_{2k} t(\beta) \cdots | \Phi_{13} \rangle$
- $\partial \longrightarrow c_1$
- $\bar{\partial} \longrightarrow \bar{c}_{-1}$

Conservation law $\bar{\partial} T_{2k+2} = \partial \Theta_{2k}$

$$\langle \Phi_{11} | \Psi_{2k+2} t(\beta) \cdots \bar{c}_{-1} | \Phi_{11} \rangle = \langle \Phi_{13} | \Psi_{2k} c_1 t(\beta) \cdots | \Phi_{13} \rangle$$

Resonance condition

$$\Delta_{\bar{\partial} T} = \Delta_{\partial \Theta} + (1 - \Delta_{\Phi})$$

Two "chiralities" and resonances II

Higher equations of motion $\frac{d}{d\alpha} D_{1n}\bar{D}_{1n}\Phi_\alpha =$? (n is odd)

$$\frac{d}{d\alpha}\langle\Phi_\alpha|D_{1n}t(\beta)\cdots\bar{D}_{1n}|\Phi_\alpha\rangle =$$
$$= \langle\Phi_{1,n+2}|t(\beta)\cdots|\Phi_{1,n+2}\rangle - \langle\Phi_{-1,n}|t(\beta)\cdots|\Phi_{-1,n}\rangle$$

Resonance condition

$$\Delta_{1n} + n = \Delta_{1,n+2} + n(1-\Delta_{13})$$

Example n=1

$$\frac{d}{d\alpha} D_{11}\bar{D}_{11}e^{\alpha\phi}\Big|_{\alpha=0} = \partial\bar{\partial}\phi = sinh(b\phi)$$

Toward a basis of physical fields

Higher modes of screening currents

- Definition

$$\psi_n = \oint \tfrac{dz}{2\pi i} z^{n-1} U(z)^{-1}$$

- Commutation relation

$$\psi_n \psi_m = -\psi_{m+2} \psi_{n-2}$$

- Intertwining property:

$$\psi_n X^{(m)} = (-1)^m X^{(m)} \psi_n$$

- Hypothesis on bound state equation for Lee-Yang $\xi = 2/3$

Specific features

Statements on space of descendants

- Left and right chiralities are not independent
- $D_{lk}\bar{D}_{lk}\Phi_{lk}$ - resonance situation: Al. Zamolodchikov higher equations of motion for a sinh-Gordon.
- Screening currents can be applied to get the higher conservation law equations
- Inverse screening currents creates operators which satisfy bound state conditions

Conclusion

Algebraic approach to descendant form factors

- Analogues of null vectors for a breaser sector of SG
- Integral representation for the Macdonald polynomials with $q = e^{-i\pi(\xi+1)}, t = e^{-i\pi\xi}$
- Analysis of resonances, conservation laws, higher equation of motion