Thermodynamics of the six-vertex model on an L-shaped domain

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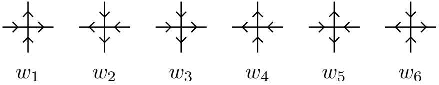
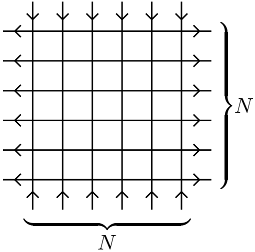
The six-vertex model with DWBC

The six-vertex model:



Domain wall boundary conditions (Korepin, 1982):



The partition function:

$$Z_N = \sum_{\mathcal{C}} \prod_{i=1}^{6} w_i^{n_i(\mathcal{C})}, \qquad \sum_{i=1}^{6} n_i(\mathcal{C}) = N^2,$$

where $n_i(\mathcal{C})$ is the number of vertices of type i in the configuration $\mathcal{C}$. The partition function is given as an $N \times N$ determinant (Izergin, 1987).

The six-vertex model with DWBC

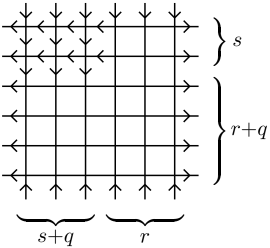
Some further results (mainly related to statistical mechanics) are:

- Thermodynamics—the free energy (Korepin, Zinn-Justin, 2000);
- Asymptotic expansion of the partition function in the thermodynamic limit (Bogoliubov, Kitaev, Zvonarev, 2002, Bleher, Fokin, Liechty, Bothner, 2006-2012);
- Boundary correlation functions (Bogoliubov, Pronko, Zvonarev, 2001, Foda, Preston, 2004, Colomo, Pronko, 2005-2006);
- Simulations of configurations—numerical evidence of the phase separation phenomena (Syljuasen, Zvonarev, 2004, Allison, Reshetikhin, 2005);
- Emptiness formation probability (CP, 2008);
- A formula for the arctic curve—the curve of spacial separation of order and disorder (CP, 2010).

There are also numerous results related to combinatorics (Kuperberg, Zeilberger, Razumov, Stroganov, Zinn-Justin, Di Francesco, ..., 1996-...).

The Emptiness Formation Probability

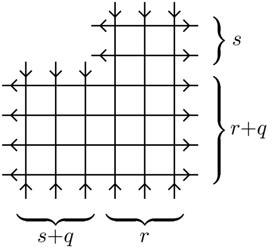
$F_{r,s,q}$ — Emptiness Formation Probability of the six-vertex model with DWBC. Non-local correlation function which describes the probability of obtaining the configuration



The EFP have been represented as an s-fold multiple integral [CP, Nucl. Phys. B 798 (2008), 340]. By making use some results from the random matrix theory it is possible to extract (in fact, to conjecture) an equation for the arctic curve [CP, J. Stat. Phys. 138 (2010), 662]. An open problem remains: Construct an asymptotic expansion of the EFP in the thermodynamic limit!

An L-shaped domain

Consider the six-vertex model on the lattice



We call it the L-shaped domain. The partition function is related to the EFP:

$$Z_{r,s,q} = \frac{Z_N}{w_2^{s(s+q)}}\, F_{r,s,q}, \qquad N = r + s + q.$$

Thus, the thermodynamics of the 6VM on the L-shaped domain as r, s, and q vary is totally controlled by the asymptotic properties of the EFP in the thermodynamic limit!

Thermodynamic limit

We are interested in the limit:

$$r, s, q \to \infty, \quad \text{with the ratios} \quad r : s : q \quad \text{fixed.}$$

In this limit, the $N \times N$ lattice is scaled onto an the unit square $[0, 1] \times [0, 1]$, with the scaled variables

$$x = \frac{s}{N}, \qquad y = \frac{s+q}{N}, \qquad x, y \in [0, 1].$$

In the thermodynamic limit the partition function of the six-vertex model on the L-shaped domain is

$$\log Z_{r,s,q} = -N^2(1 - xy) f(x, y) + o(N^2)$$

where $f(x, y)$ is the free energy per site. We expect that the EFP behaves as:

$$\log F_{r,s,q} = -N^2 \sigma(x, y) + o(N^2),$$

and so the functions $\sigma(x, y)$ and $f(x, y)$ are related by

$$(1 - xy) f(x, y) = f(0, 0) + xy \log w_2 + \sigma(x, y).$$

The free-fermion point

We choose the weights as

$$w_1 = w_2 = \sqrt{1-\alpha}, \qquad w_3 = w_4 = \sqrt{\alpha}, \qquad w_5 = w_6 = 1, \qquad \alpha \in [0,1].$$

The multiple integral representation for the EFP in this case is

$$F_{r,s,q} = \frac{(-1)^{s(s+1)/2}}{s!} \oint_{C_0} \cdots \oint_{C_0} \prod_{j<k} (z_j - z_k)^2 \prod_{j=1}^{s} \frac{(\alpha z_j + 1 - \alpha)^{r+q}}{z_j^r (z_j - 1)^s} \frac{\mathrm{d}z^s}{(2\pi\mathrm{i})^s}.$$

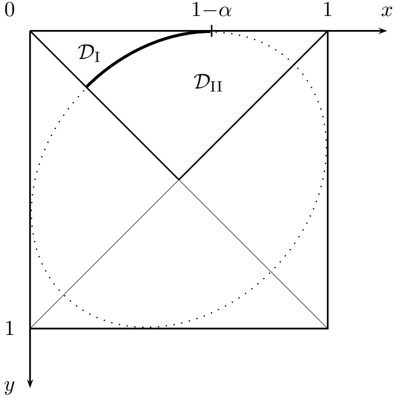
It admits various equivalents representations (Hankel determinants, Fredholm determinants, Toda chain solutions) [AP, J. Math. Sci. 192 (2013), 101]. In particular,

$$F_{r,s,q} = \frac{(q!)^s}{\prod_{k=0}^{s-1} (q+k)!k!} \frac{(1-\alpha)^{s(s+q)}}{\alpha^{s(s-1)/2}} \det_{1\le j,k \le s} \left[\sum_{m=0}^{r-1} m^{j+k-2} \binom{m+q}{m} \alpha^m \right].$$

This formula is valid for $q \ge 0$ and allows one to find $\sigma(x,y)$ for $x \ge y$.

In a different form and in different notations it appeared previously in the random grows models context [Johansson, 2000].

The Arctic Ellipse and domains $\mathcal{D}_{\mathrm{I}}$, $\mathcal{D}_{\mathrm{II}}$



Dotted line: the Arctic Ellipse—the phase separation curve of the six-vertex model with DWBC (the free-fermion weights),

$$\frac{(1-x-y)^2}{\alpha}+\frac{(x-y)^2}{1-\alpha}=1.$$

The main result

We obtain

$$\sigma(x,y)=0, \qquad (x,y)\in\mathcal{D}_{\mathrm{I}},$$

and

$$\begin{aligned}\sigma(x,y)=&\,xy\log\frac{h}{\eta}-\frac{(1-x-y)^2}{2}\log\frac{1-h}{1-\eta}-\frac{1}{2}\log\frac{1+h}{1+\eta}\\&+(1-x)y\log\frac{y+(1-x)h}{y+(1-x)\eta}+x(1-y)\log\frac{x+(1-y)h}{x+(1-y)\eta}\\&-(1-x)x\log\frac{x+(1-x)h}{x+(1-x)\eta}-(1-y)y\log\frac{y+(1-y)h}{y+(1-y)\eta}\\&-\frac{(x-y)^2}{2}\log\frac{x+y+(2-x-y)h}{x+y+(2-x-y)\eta}, \qquad (x,y)\in\mathcal{D}_{\mathrm{II}},\end{aligned}$$

where

$$h=h(x,y):=\sqrt{\frac{xy}{(1-x)(1-y)}}$$

and $\eta=\eta(x,y;\alpha)$ is such that

$$\eta\in[0,1], \qquad \alpha\frac{(1+\eta)^2\big(x+(1-x)\eta\big)\big(y+(1-y)\eta\big)}{(1-\eta)^2\big(y+(1-x)\eta\big)\big(x+(1-y)\eta\big)}=1.$$

Third-order phase transition

Near the Arctic Ellipse the function $\sigma(x, y)$ vanishes as ϵ^3, where ϵ is the distance to the Arctic Ellipse (from the interior of the ellipse).

This can seen differently, as a phase transition in the parameter α:

$$\sigma(x, y) = \begin{cases} 0 & \alpha \leq \alpha_\mathrm{c} \\ C(\alpha - \alpha_\mathrm{c})^3 + O\left((\alpha - \alpha_\mathrm{c})^4\right) & \alpha \geq \alpha_\mathrm{c} \end{cases}$$

where $C = C(x, y, \alpha) > 0$ and the critical value $\alpha_\mathrm{c} = \alpha_\mathrm{c}(x, y)$ is the value of α such that the given point (x, y) is a point of the top left portion of the Arctic Ellipse,

$$\alpha_\mathrm{c} = \left(\sqrt{(1-x)(1-y)} - \sqrt{xy}\right)^2 .$$

EFP as a matrix model

To obtain $\sigma(x,y)$ above, we start with the Hankel determinant formula and write the EFP as

$$F_{r,s,q} = \frac{(1-\alpha)^{s(s+q)}}{\alpha^{s(s+1)/2}} I_{r,s,q}$$

where $I_{r,s,q}$ is a random matrix model integral (with the discrete measure):

$$I_{r,s,q} = \frac{1}{s!} \prod_{j=0}^{s-1} \frac{q!}{j!(j+q)!} \sum_{m_1=0}^{r-1} \cdots \sum_{m_s=0}^{r-1} \prod_{j<k} (m_j - m_k)^2 \prod_{j=1}^{s} \binom{q+m_j}{q} \alpha^{m_j}.$$

In the thermodynamic limit s, r, and q are large, and we have

$$I_{r,s,q} = \exp\left\{s^2 \Phi + o(s^2)\right\}, \qquad s \to \infty,$$

where $\Phi = \Phi(R, Q, \alpha)$ and the parameters $R \in [1, \infty)$ and $Q \in [0, \infty)$ are

$$R := \frac{r}{s} = \frac{1-x}{y}, \qquad Q := \frac{q}{s} = \frac{x-y}{y}.$$

EFP as a matrix model

The standard approach of random matrix theory: the sums are approximated by integrals, by introducing the rescaled variables μ_j:

$$m_j = s\mu_j, \qquad j = 1, \ldots, s.$$

So, as $s \to \infty$,

$$I_{r,s,q} \propto \int_0^R \cdots \int_0^R \prod_{1 \le j < k \le s} (\mu_k - \mu_k)^2 \exp\left\{ -s \sum_{j=1}^{s} V(\mu_j) \right\} \mathrm{d}M_s(\mu).$$

The potential $V(\mu)$ is

$$V(\mu) := -\lim_{s \to \infty} \frac{1}{s} \log\left(\binom{q + s\mu}{q} \alpha^{s\mu} \right), \qquad q = Qs.$$

The integration measure $\mathrm{d}M_s(\mu)$ is

$$\mathrm{d}M_s(\mu) = \prod_{1 \le j < k \le s} H\left(|\mu_j - \mu_k| - s^{-1}\right) \mathrm{d}^s\mu.$$

The product of the Heaviside-functions $\prod_{j<k} H\left(|\mu_j - \mu_k| - s^{-1}\right)$ takes into account the discreteness of the original variables.

The resolvent

In the limit $s \to \infty$ the solution of the saddle-point problem is described by the resolvent

$$W(z) = \int_S \frac{\rho(\mu)}{z - \mu}\, \mathrm{d}\mu, \qquad z \notin S,$$

where $\rho(\mu)$ is the density, and S is the support of the density, $S \subseteq [0, R]$. Given a resolvent $W(x)$,

$$\rho(\mu) = -\frac{1}{2\pi \mathrm{i}} \left[W(\mu + \mathrm{i}0) - W(\mu - \mathrm{i}0) \right], \qquad \mu \in S.$$

The density, by definition, is subject to the normalization condition

$$\int_S \rho(\mu)\, \mathrm{d}\mu = 1,$$

In the case of discrete variables it must satisfy the additional condition [Douglas, Kazakov, 1993, Brezin, Kazakov, 2000]:

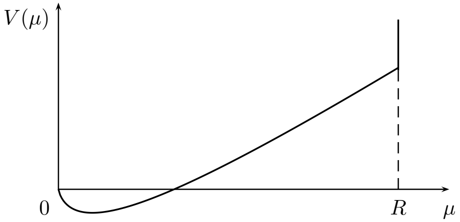
$$\rho(\mu) \leq 1, \qquad \mu \in S.$$

Solutions tend to accumulate near the bottom of the well of the potential, and saturated regions in S, where $\rho(\mu) = 1$, may arise.

The one-band ansatz

The potential of our RMM is

$$V(\mu) = -\mu \log \alpha + \mu \log \mu - (\mu + Q) \log(\mu + Q) + Q \log Q, \quad \mu \in [0, R].$$



The function $V(\mu)$ has the following properties:

- It has a single minimum, at the point $\mu = \alpha Q/(1-\alpha)$;
- As $\mu \to \infty$, it is linear, with a positive slope, $V(\mu) \sim (-\log \alpha)\mu$;
- The allowed values of μ are in the interval $[0, R]$

This implies that $\rho(\mu)$ takes intermediate values on a subinterval $[a, b]$, where a and b to be determined, while on the subintervals $[0, a]$ and $[b, R]$ it is equal to 0 or 1. This is the so-called one-band ansatz [Baik, Kriecherbauer, McLaughlin, Miller, Discrete Orthogonal Polynomials: Asymptotics and Applications, 2007].

The resolvent

The resolvent is given by

$$W(z) = H(z) + \int_{S_{\rho=1}} \frac{1}{z-\mu}\,\mathrm{d}\mu, \qquad H(z) = \int_{S_{\rho<1}} \frac{\rho(\mu)}{z-\mu}\,\mathrm{d}\mu.$$

The function $H(z)$ satisfies

$$H(\mu + \mathrm{i}0) + H(\mu - \mathrm{i}0) = U(\mu), \qquad \mu \in S_{\rho<1},$$

where

$$U(z) = -2\int_{S_{\rho=1}} \frac{1}{z-\mu}\,\mathrm{d}\mu + V'(z).$$

In the case of the one-band ansatz $S_{\rho<1} = [a,b]$ and the function $H(z)$ is

$$H(z) = \frac{\sqrt{(z-a)(z-b)}}{2\pi}\int_a^b \frac{U(\mu)}{(z-\mu)\sqrt{(\mu-a)(b-\mu)}}\,\mathrm{d}\mu.$$

The end-points a and b can be found by solving the equations arising from the condition that $W(z) \sim 1/z$ as $s \to \infty$.

The free energy

Define the first moment of the density:

$$E = \int_S \mu\rho(\mu)\,\mathrm{d}\mu.$$

It can be found from the resolvent $W(z)$, by noting that

$$W(z) = \frac{1}{z} + \frac{E}{z^2} + O(z^{-3}), \qquad |z| \to \infty.$$

The quantity Φ (the "free energy" of the matrix model) satisfies

$$\frac{\partial}{\partial \log \alpha}\Phi = E.$$

and hence

$$\Phi = \int E\,\frac{\mathrm{d}\alpha}{\alpha} + C.$$

where $C = C(R, Q)$ is some function independent of α. It is fixed by noting that the Hankel determinant for the EFP admits exact evaluation for $\alpha = 0$ and as $\alpha \to 1$.

Two regimes

We obtain that the first moment of the density E (hence, Φ) has two different expressions depending on either $R > R_\mathrm{c}$ or $R < R_\mathrm{c}$, where

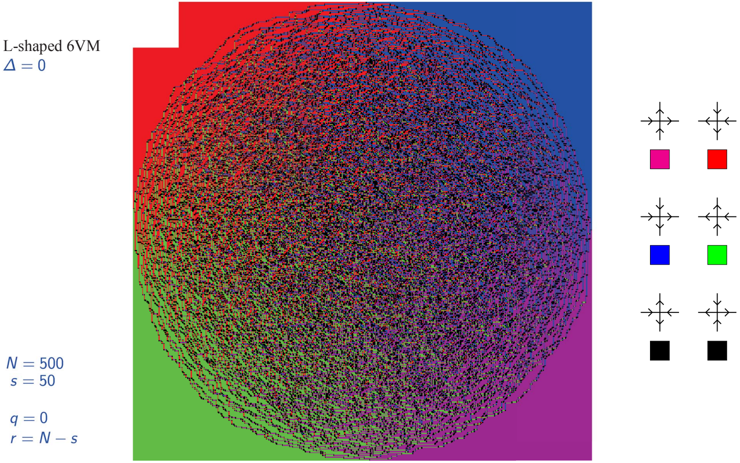
$$R_\mathrm{c} = \frac{\left(1 + \sqrt{\alpha(1+Q)}\right)^2}{1-\alpha}, \qquad Q \in [0, \infty).$$

The first case corresponds to zero density in the interval $[b, R]$; the points $(x, y) = (x(R, Q), y(R, Q))$ take their values in the region $\mathcal{D}_\mathrm{I}$.

The second case corresponds to $\rho(\mu) = 1$ in the interval $[b, R]$; the points $(x, y) = (x(R, Q), y(R, Q))$ take their values in the region $\mathcal{D}_\mathrm{II}$.

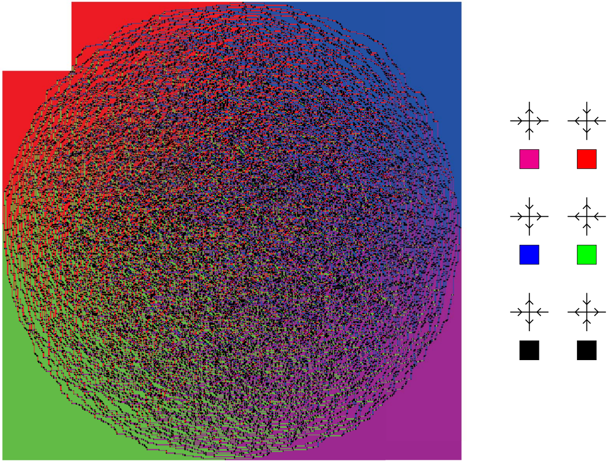
The value of R_c corresponds to $b = R$.

Some numerical results, $\alpha = 1/2$ [Colomo, Sportiello, 2014]



Some numerical results, $\alpha = 1/2$ [Colomo, Sportiello, 2014]

L-shaped 6VM
$\Delta = 0$

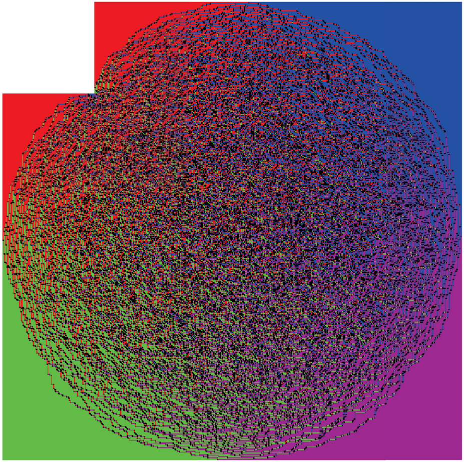


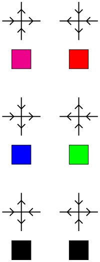
$N = 500$
$s = 75$

$q = 0$
$r = N - s$

Some numerical results, $\alpha = 1/2$ [Colomo, Sportiello, 2014]

L-shaped 6VM
$\Delta = 0$



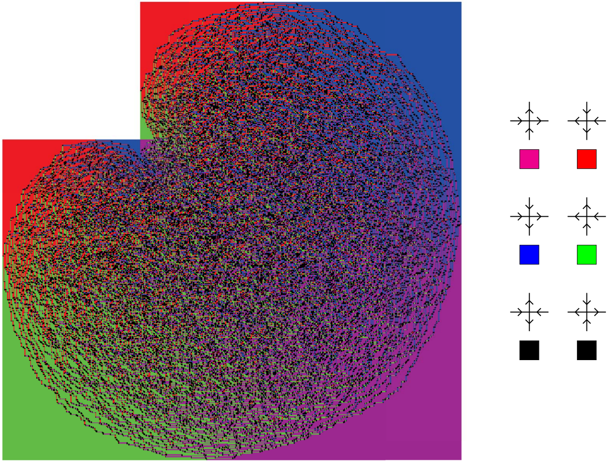


$N = 500$
$s = 100$

$q = 0$
$r = N - s$

Some numerical results, $\alpha = 1/2$ [Colomo, Sportiello, 2014]

L-shaped 6VM
$\Delta = 0$

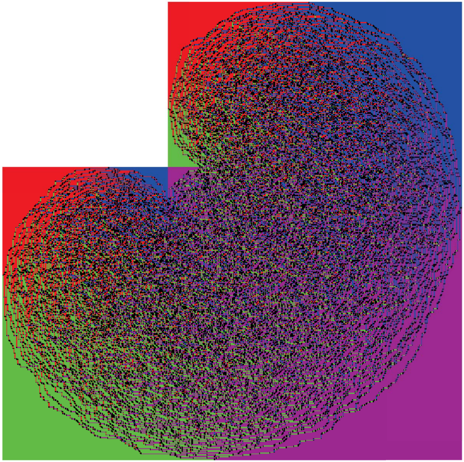


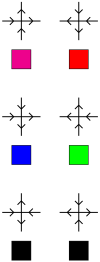
$N = 500$
$s = 150$

$q = 0$
$r = N - s$

Some numerical results, $\alpha = 1/2$ [Colomo, Sportiello, 2014]

L-shaped 6VM
$\Delta = 0$



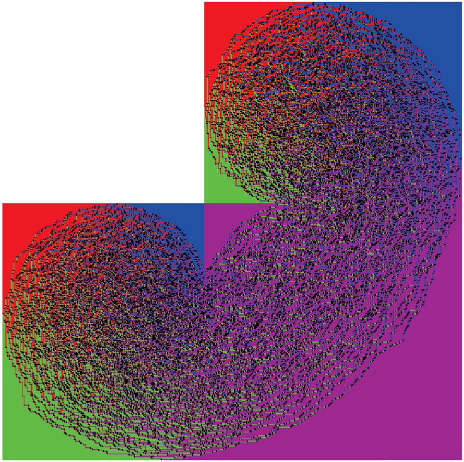


$N = 500$
$s = 180$

$q = 0$
$r = N - s$

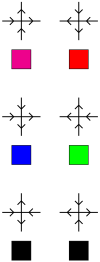
Some numerical results, $\alpha = 1/2$ [Colomo, Sportiello, 2014]

L-shaped 6VM
$\Delta = 0$



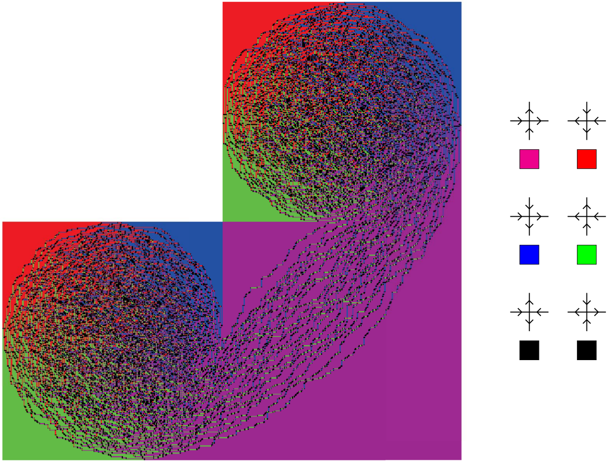
$N = 500$
$s = 220$

$q = 0$
$r = N - s$



Some numerical results, $\alpha = 1/2$ [Colomo, Sportiello, 2014]

L-shaped 6VM
$\Delta = 0$

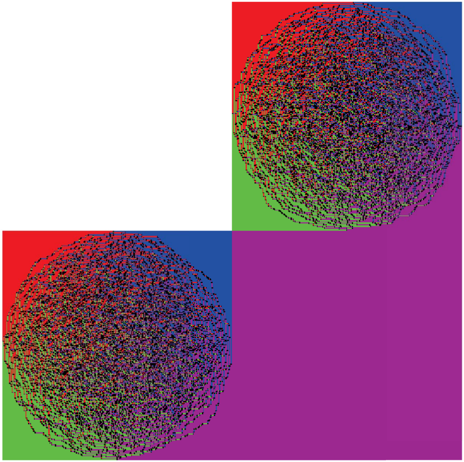


$N = 500$
$s = 240$

$q = 0$
$r = N - s$

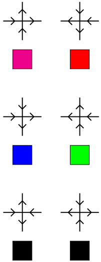
Some numerical results, $\alpha = 1/2$ [Colomo, Sportiello, 2014]

L-shaped 6VM
$\Delta = 0$



$N = 500$
$s = 250$

$q = 0$
$r = N - s$



Open problems

- Construct asymptotic expansion for the partition function $Z_{r,s,q}$ of the 6VM on the L-shaped domain in the thermodynamic limit, extending the results of Bleher, Fokin, Liechty, Bothner for the partition function Z_N of the 6VM with DWBC.
- Derive the arctic curve of the 6VM on the L-shaped domain.
- Try to extend any of the results about the EFP out the free-fermion point.