Form factor approach to correlation functions in massless quantum integrable models.

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Form factor approach to the asymptotic behavior of correlation functions in critical models, N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. Slavnov and V. Terras, J. Stat. Mech. (2011).

Form factor approach to dynamical correlation functions in critical models, N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. Slavnov and V. Terras, J. Stat. Mech. (2012).

Long-distance asymptotic behavior of multi-point correlation functions in massless quantum integrable models, N. Kitanine, K. K. Kozlowski, J. M. Maillet and V. Terras, J. Stat. Mech. (2014).

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Outline

Some integrable models we will consider

- The XXZ spin-1/2 chain

$$\mathcal{H}_{XXZ} = J \sum_{n=1}^{L} \left\{ \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z \; + \; h \sigma_n^z \right\} \quad , \quad \sigma_{n+L} \equiv \sigma_n$$

L: length of circle, Δ anisotropy parameter, $h > 0$ magnetic field.

  - Coordinate Bethe Ansatz for the XXX chain $\Delta = 1$ (**'31** Bethe)

- The non-linear Schrödinger model

$$H = \int_0^L \left\{ \partial_y \Psi^\dagger(y) \, \partial_y \Psi(y) + c \Psi^\dagger(y) \, \Psi^\dagger(y) \, \Psi(y) \, \Psi(y) - h \Psi^\dagger(y) \, \Psi(y) \right\} \mathrm{d}y$$

L: length of circle, $c > 0$ coupling constant (repulsive regime), $h > 0$ chemical potential.

  - Eigenfunctions and spectrum (**'63** Lieb, Liniger).

$$\mathrm{e}^{iL\lambda_j} = \prod_{\substack{a=1 \\ \neq j}}^{N} \frac{\lambda_j - \lambda_a + ic}{\lambda_j - \lambda_a - ic} \qquad \text{so that} \qquad H|\{\lambda_j\}\rangle = \left(\sum_{k=1}^{N} \lambda_k^2 - h \right) |\{\lambda_j\}\rangle$$

Low-lying excitations in 1D quantum Hamiltonians

- 1D *gapless* models at $T = 0K$ are critical
  - ★ **'70** Polyakov Conformal invariance of correlation functions in long-distance regime ;
  - ★ **'84** Cardy Central charge ⇝ finite-size corrections to ground state energy ;

$$E_{G.S.} = L\varepsilon - c\frac{\pi v_F}{6L} + \mathrm{O}\left(\frac{1}{L^2}\right) \quad \text{and} \quad E_{\text{ex}} - E_{G.S.} = \frac{2\pi v_F}{L}\delta$$

  - ★ Bethe Ansatz ⇝ spectrum given by solutions to algebraic equations

$$F(\lambda_j) = \prod_{a=1}^{N} S(\lambda_j, \lambda_k) \quad \text{and} \quad E(\{\lambda_j\}) = \sum_{j=1}^{N} \varepsilon_0(\lambda_j)$$

  - ★ Methods for computing finite-size corrections from Bethe Ansatz
    '87-'95 (Batchelor, Destri, DeVega, Klumper, Pearce, Woynarowich, Zittartz , ...) ;

- ⊛ Proof of Cardy's predictions for the conformal structure of spectrum:

$$c = 1 \qquad \delta = \left(\frac{N_1}{2\mathcal{Z}}\right)^2 + (\mathcal{Z}N_2)^2 + N_3 \quad \text{and} \quad \text{linear integral equations} \rightsquigarrow v_F\,,\ \mathcal{Z}$$

Asymptotic behavior of correlation functions

- Critical model $\rightsquigarrow$ algebraic in distance decay of correlators.
  - ⋆ **'75** Luther, Peschel , **'81** Haldane Luttinger liquid approach to asymptotics ;
  - ⋆ **'84** Cardy Central charge, scaling dimensions $\rightsquigarrow$ CFT approach to asymptotics;

$\Rightarrow$ Predictions of critical exponents by correspondence with Luttinger liquid/CFT.

- NLSM ≡ quantum critical model at $T = 0K$ $\rightsquigarrow$ density operator $j(x) = \Psi^{\dagger}(x)\,\Psi(x)$

$$\frac{\langle G.S.|j(x)\,j(0)\,|G.S.\rangle}{\langle G.S.\,|\,G.S.\rangle} = \langle j(x)\,j(0)\rangle \simeq \langle j(0)\rangle^2 + \frac{C_1}{x^2} + C_2\,\frac{\cos(2xp_F)}{x^{2\mathcal{Z}^2}} + \ldots$$

and $$\langle \Psi(x)\,\Psi^{\dagger}(0)\rangle \simeq C_3\, x^{-\frac{1}{2\mathcal{Z}^2}} + \ldots$$

No access to non universal constants C_k.

Conjecture for C_k in XXZ at zero magnetic field **'99** Lukyanov , **'03** Lukyanov,Terras .

Turning the time on

- Predictions for the long-distance/long-time behavior at $T = 0K$ restricted to $x \gg v_F t$:

$$\left\langle j(x,t)\, j(0,0) \right\rangle \simeq \langle j(0,0)\rangle^2 + C_1' \frac{x^2 + v_F^2 t^2}{\left(x^2 - v_F^2 t^2\right)^2} + C_2' \frac{cos\,(2x p_F)}{\left(x^2 - v_F^2 x^2\right)^{\mathcal{Z}^2}} + \dots$$

$\Rightarrow$ *Consistency problem* with time-dependent asymptotics

$$\frac{x^2 + v_F^2 t^2}{\left(x^2 - v_F^2 t^2\right)^2}\left(1 + \mathrm{o}(1)\right) = \frac{1}{x^2}\left(1 + \mathrm{o}(1)\right) \quad \text{when} \;\; x \gg v_F t$$

- What happens when x and $v_F t$ are of the same order asymptotically?

The edge exponents for dynamical structure factors

- Experiments measure dynamical structure factors (Fourier transforms)

$$S(k,\omega) = \int_{\mathbb{R}^2} \mathrm{e}^{i(\omega t - kx)} \langle j(x,t)\, j(0,0)\rangle \mathrm{d}x\mathrm{d}t$$

  ⇝ DSF measured by Fourier sampling of time of flight images or Bragg spectroscopy.
  ⇝ Predictions for the behavior near the edges

★ **'67** (Mahan), **'67** (Nozières, De Dominicis) Arguments for a power-law behavior near edges.

★ **'08** (Glazman, Imambekov) Non-linear Luttinger liquid ⇝ predictions for edge exponents.

$$S(k,\omega) \simeq \mathscr{A}(k) \cdot \xi\left(\omega - \varepsilon_h(k)\right) \cdot \left[\omega - \varepsilon_h(k)\right]^{\vartheta}$$

★ **'09** (Affleck, Pereira, White) X-ray edge-type model ⇝ predictions for edge exponents.

★ **'10** (Caux, Glazman, Imambekov, Shashi) Predictions for $\mathscr{A}(k)$ (NLSM);

- Can these predictions be confirmed by a computation from the microscopic model?

Results for XXZ from multiple integrals and RHP

'09 Kitanine, Kozlowski, M., Slavnov, Terras

Generating function

$Q^{\kappa}_{1,m} = \prod_{n=1}^{m} \left(\frac{1+\kappa}{2} + \frac{1-\kappa}{2} \cdot \sigma^z_n \right)$

Asymptotic behavior

$$\langle e^{\beta Q_{1m}} \rangle = \underbrace{G^{(0)}(\beta, m)[1 + o(1)]}_{\text{non-oscillating terms}} + \underbrace{\sum_{\sigma=\pm} G^{(0)}(\beta + 2i\pi\sigma, m)[1 + o(1)]}_{\text{oscillating terms}}$$

$$G^{(0)}(\beta, m) = C(\beta)\, e^{m\beta D}\, m^{\frac{\beta^2}{2\pi^2} Z(q)^2}$$

- $Z(\lambda)$ is the dressed charge $\quad Z(\lambda) + \int_{-q}^{q} \frac{\mathrm{d}\mu}{2\pi} K(\lambda - \mu)\, Z(\mu) = 1$
- D is the average density $\quad D = \int_{-q}^{q} \rho(\mu) \mathrm{d}\mu = \frac{1 - \langle \sigma^z \rangle}{2} = \frac{p_F}{\pi}$
- The coefficient $C(\beta)$ is given as the ratio of four Fredholm determinants.
- sub-leading oscillating terms restore the $2\pi i$-periodicity in β

Amplitudes from form factors

2-point function asymptotic behavior

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle = (2D-1)^2 - \frac{2Z(q)^2}{\pi^2 m^2} + 2|F_{\sigma^z}|^2 \cdot \frac{\cos(2mp_F)}{m^{2Z(q)^2}} + o\left(\frac{1}{m^2}, \frac{1}{m^{2Z(q)^2}}\right)$$

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The umklapp form factor

$$\lim_{N,M\to\infty} \left(\frac{M}{2\pi}\right)^{2\mathcal{Z}^2} \frac{|\langle \psi(\{\mu\})|\sigma^z|\psi(\{\lambda\})\rangle|^2}{\|\psi(\{\mu\})\|^2 \cdot \|\psi(\{\lambda\})\|^2} = |F_{\sigma^z}|^2.$$

with

$$2\mathcal{Z}^2 = Z(q)^2 + Z(-q)^2$$

- $\{\lambda\}$ are the Bethe parameters of the ground state
- $\{\mu\}$ are the Bethe parameters for the excited state with one particle and one hole on opposite sides of the Fermi boundary (umklapp type excitation).

Amplitudes from form factors

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$\hookrightarrow$ Higher terms in the asymptotic expansion will involve n - particle/holes form factors corresponding to $2np_F$ oscillations
$\hookrightarrow$ Properly normalized form factors will be related to the corresponding amplitudes

$\hookrightarrow$ It strongly suggests to analyze the asymptotic behavior of the correlation function directly from the form factor series!

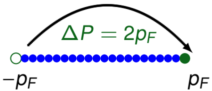
Long-distance asymptotics of densities at $T = 0K$

'11 Kitanine, Kozlowski, M., Slavnov, Terras

spin-spin correlation function of the XXZ chain at $T = 0K$:

$$\frac{\langle G.S. | \sigma_1^z \sigma_m^z | G.S. \rangle}{\langle G.S. | G.S. \rangle} = \langle \sigma^z \rangle^2 - \frac{2\mathcal{Z}^2}{\pi^2 m^2} (1 + \mathrm{o}(1)) + \sum_{\ell=1}^{+\infty} \frac{2\cos(2m\ell p_F)}{m^{2\ell^2 \mathcal{Z}^2}} \cdot |\mathcal{F}_\ell|^2 (1 + \mathrm{o}(1))$$

$$|\mathcal{F}_\ell|^2 = \lim_{L \to +\infty} \left(\frac{L}{2\pi} \right)^{2\ell^2 \mathcal{Z}^2} \frac{\left| \langle G.S. | \sigma_1^z | \mathrm{umkp} - \ell \rangle \right|^2}{\| G.S. \|^2 \cdot \| \mathrm{umkp} - \ell \|^2}$$



- ⋆ ground state in positive chemical potential
- ⋆ one Umklapp excitation $\Delta E = 0$ $\Delta P = 2p_F$.

✔ Confirms CFT and Luttinger liquid predictions.

✔ Agrees with RHP analysis of the multiple integrals representations (**'09** KKMST).

✔ Similar results for NLSM.

T=0K leading harmonics in long-time & distance asymptotics

'12 Kitanine, Kozlowski, M., Slavnov, Terras

Currents: $j(x,t) \equiv \mathrm{e}^{iHt}\Psi^{\dagger}(x)\Psi(x)\mathrm{e}^{-iHt}$ asymptotic regime $x \to +\infty$ and x/t fixed.

Overall structure of the asymptotic series (space-like regime) :

$$\left\langle j(x,t)\, j(0,0)\right\rangle = \left(\frac{p_F}{\pi}\right)^2 - \frac{\mathcal{Z}^2}{2\pi^2}\frac{x^2+t^2v_F^2}{\left(x^2-t^2v_F^2\right)^2}\left(1+\mathrm{o}(1)\right)$$

$$+ \sum_{\substack{\ell_+;\ell_-\in\mathbb{Z} \\ \ell_++\ell_-\le 0}}^{*} \frac{\mathrm{e}^{ix\ell_+p_F}}{\left[-i(x-v_Ft)\right]^{\Delta^{(R)}_{\ell_+;\ell_-}}}\,\frac{\mathrm{e}^{-ix\ell_-p_F}}{\left[i(x+v_Ft)\right]^{\Delta^{(L)}_{\ell_+;\ell_-}}}$$

$$\times\, \mathrm{e}^{-i(\ell_++\ell_-)[xp(\lambda_0)-t\varepsilon(\lambda_0)]}\left(\frac{[p'(\lambda_0)]^2}{-i[xp''(\lambda_0)-t\varepsilon''(\lambda_0)]}\right)^{\frac{|\ell_++\ell_-|^2}{2}} \cdot \frac{(2\pi)^{\frac{|\ell_++\ell_-|}{2}}\left|\mathcal{F}^{(j)}_{\ell_+,\ell_-}\right|^2}{G\left(1+|\ell_++\ell_-|\right)}\left(1+\mathrm{o}(1)\right)\,.$$

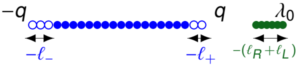
★ λ_0 **Saddle-point** of the oscillating phase: $p'(\lambda_0) - t\varepsilon'(\lambda_0)/x = 0$.

⇝ p dressed momentum & ε dressed energy.

Form factor interpretation of the amplitudes

$$\left|\mathcal{F}^{(j)}_{\ell_+,\ell_-}\right|^2 = \lim_{L\to+\infty}\left\{\left(\frac{L}{2\pi}\right)^{|\ell_++\ell_-|^2+\Delta^{(R)}_{\ell_+;\ell_-}+\Delta^{(L)}_{\ell_+;\ell_-}} \cdot \frac{\left|\left\langle G.S.\left| j(0) \right| \mathrm{Ex}(\ell_+;\ell_-)\right\rangle\right|^2}{\left\|G.S.\right\|^2 \cdot \left\|\mathrm{Ex}(\ell_+;\ell_-)\right\|^2}\right\}$$

★ ℓ_+: # additional particles at q ℓ_-: # additional particles at $-q$ $|\ell_+ + \ell_-|$: # particles at λ_0



★ ground state in positive chemical potential

★ excitation $\begin{cases} \Delta E = |\ell_+ + \ell_-|\varepsilon(\lambda_0) \\ \Delta P = |\ell_+ + \ell_-|p(\lambda_0) - |\ell_+|p_F - |\ell_-|(-p_F) \end{cases}$

- Critical exponents $\Delta^{(R/L)}_{\ell_+;\ell_-}$ originate from excitations on Fermi boundaries.

$$\Delta^{(R)}_{\ell_+;\ell_-} = (\ell_+ + \ell_-)\phi(q,\lambda_0) - \ell_-\phi(q,-q) - \ell_+\phi(q,q) \qquad \left(I - \frac{K}{2\pi}\right)\cdot\phi(\lambda,\mu) = \theta(\lambda-\mu)$$

- Critical exponent $\frac{|\ell_+ + \ell_-|^2}{2}$ originates from gaussian saddle-point.

✔ Agrees with the first terms obtained through Natte series (**'11** Kozlowski, Terras).

The power-law behavior of dynamical structure factors (NLSM)

'12 Kitanine, Kozlowski, M., Slavnov, Terras

(k,ω) configuartion close to the hole excitation line

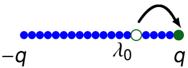
$$(p_F - p(\lambda_0)\,,-\varepsilon(\lambda_0)) \qquad \text{with} \qquad \lambda_0 \in \,]-q\,;q\,[\ .$$

★ The *hole* treshold

$$S\big(p_F - p(\lambda_0), -\varepsilon(\lambda_0) + \delta\omega\big) \simeq \frac{\xi\,(\delta\omega)\,[\delta\omega]^{\Delta^{(R)}_{1;0}\,+\,\Delta^{(L)}_{1;0}-1}}{[v+v_F]^{\Delta^{(R)}_{1;0}}\,[v_F - v]^{\Delta^{(L)}_{1;0}}} \cdot \frac{(2\pi)^2\,\big|\mathcal{F}^{(j)}_{1,0}\big|^2}{\Gamma(\Delta^{(R)}_{1;0} + \Delta^{(L)}_{1;0})}\ .$$

★ v: velocity of the hole at λ_0 $\qquad$ v_F: velocity excitations on Fermi boundary.

$$\big|\mathcal{F}^{(j)}_{1,0}\big|^2 = \lim_{L\to+\infty}\left\{\left(\frac{L}{2\pi}\right)^{1+\Delta^{(R)}_{1;0}\,+\,\Delta^{(L)}_{1;0}} \frac{\big|\big\langle G.S.\big|\,j\,(0)\,\big|\mathrm{Ex}\big\rangle\big|^2}{\big\|G.S.\big\|^2\cdot\big\|\mathrm{Ex}\big\|^2}\right\}$$



★ ground state

★ excitation $\left\{\begin{array}{lcl}\Delta E & = & -\varepsilon\,(\lambda_0) \\ \Delta P & = & p_F - p(\lambda_0)\end{array}\right.$

(k,ω) configuration close to the particle excitation line

$$(p(\lambda_0)-p_F,\varepsilon(\lambda_0)) \qquad \text{with} \qquad \lambda_0 \in \,]\,q\,;+\infty\,[\ .$$

★ The *particle* treshold

$$S\big(p(\lambda_0)-p_F,\varepsilon(\lambda_0)+\delta\omega\big) \simeq \frac{[\delta\omega]^{\Delta^{(R)}_{-1;0}+\Delta^{(L)}_{-1;0}-1}}{[v+v_F]^{\Delta^{(R)}_{-1;0}}[v_F-v]^{\Delta^{(L)}_{-1;0}}}\cdot\frac{(2\pi)^2\left|\mathcal{F}^{(j)}_{-1,0}\right|^2}{\Gamma(\Delta^{(R)}_{1;0}+\Delta^{(L)}_{1;0})}$$
$$\times\ \frac{\xi(\delta\omega)\sin\left[\pi\Delta^{(L)}_{-1;0}\right]+\xi(-\delta\omega)\sin\left[\pi\Delta^{(R)}_{-1;0}\right]}{\sin\pi\left[\Delta^{(R)}_{-1;0}+\Delta^{(L)}_{-1;0}\right]}$$

✔ Microscopic model approach ⇝ the non-linear Luttinger-based predictions.

The form factor approach

Form factor expansion for finite L of $\mathcal{O}(x,t) \equiv e^{iHt}\mathcal{O}(x)\,e^{-iHt}$

$$\begin{aligned}\langle\, G.S.\,|\mathcal{O}(x,t)\,\mathcal{O}^{\dagger}(0,0)\,|\,G.S.\,\rangle &= \sum_{\{\mu\}_{\rm ex}} \langle\, G.S.\,|e^{-ixP+itH}\mathcal{O}(0,0)\,e^{ixP-itH}|\,\{\mu\}_{\rm ex}\,\rangle\langle\,\{\mu\}_{\rm ex}\,|\mathcal{O}^{\dagger}(0,0)\,|\,G.S.\,\rangle \\ &= \sum_{\{\mu\}_{\rm ex}} e^{ix(P_{\rm G.S.}-P_{\rm ex})-it(\mathcal{E}_{G.S.}-\mathcal{E}_{\rm ex})}\left|\langle\, G.S.\,|\mathcal{O}(0,0)\,|\,\{\mu\}_{\rm ex}\,\rangle\right|^2\end{aligned}$$

Steps of the computation

- Characterize the excitations above the ground state;
- Asymptotic in *size* L formula for $\langle\, G.S.\,|\mathcal{O}(0,0)|\,\{\mu\}_{\rm ex}\,\rangle$;
- Localize sums : at saddle-points and ends of Fermi zone ;
- Sum-up in the asymptotic regime.

Free fermion model in finite volume

- Eigenfunctions $\rightsquigarrow$ from plane-waves $\varphi(\boldsymbol{x} \mid \{\lambda_a\}_1^N) = \exp\left\{i \sum_{k=1}^{N} \lambda_k x_k\right\}$
- Boundary conditions λ_a $\rightsquigarrow$ quantization of momenta $\lambda_a = \frac{2\pi}{L} n_a$ for some integers n_a .
- Simple form of spectrum $\mathcal{E}(\{\lambda_a\}_1^N) = \sum_{a=1}^{N} \lambda_a^2$ and $\mathcal{P}(\{\lambda_a\}_1^N) = \sum_{a=1}^{N} \lambda_a$

Ground state Momenta tightly packed around origin $\rightsquigarrow$ $n_a = a - (N+1)/2$
Particle-hole excitations $\rightsquigarrow$ other choices of integers:

$$n_j = j - \frac{N+1}{2} \text{ for } j \in \{1, \ldots, N\} \setminus \{h_1, \ldots, h_n\} \quad \text{and} \quad n_{h_a} = p_a - \frac{N+1}{2} \text{ for } a \in \{1, \ldots, n\}$$

- "holes" in continuous distribution of rapidities at $\mu_{h_1}, \ldots, \mu_{h_n}$
- new "particle" rapidities at $\mu_{p_1}, \ldots, \mu_{p_n}$

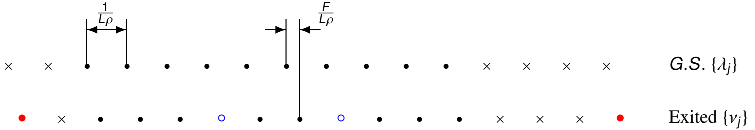
$\Rightarrow$ Excitation spectrum is additive.

$$\mathcal{P}_{\text{ex}} - \mathcal{P}_{G.S.} = \sum_{a=1}^{n} \mu_{p_a} - \mu_{h_a} \quad \text{and} \quad \mathcal{E}_{\text{ex}} - \mathcal{E}_{G.S.} = \sum_{a=1}^{n} \mu_{p_a}^2 - \mu_{h_a}^2$$

Excited states in the interacting case

Particle-hole excitations

- "holes" in continuous distribution of rapidities at $\mu_{h_1}, \ldots, \mu_{h_n}$
- new "particle" rapidities at $\mu_{p_1}, \ldots, \mu_{p_n}$



$\Rightarrow$ Excited state's rapidities ν_j shifted infinitesimally in respect to GS rapidities λ_j.

$$\nu_j - \lambda_j = \frac{1}{L\rho(\lambda_j)} \cdot F\left(\lambda_j \,\middle|\, \begin{matrix} \mu_{p_1}, \ldots, \mu_{p_n} \\ \mu_{h_1}, \ldots, \mu_{h_n} \end{matrix}\right) + \mathrm{O}(L^{-2}) \qquad j \in \{1, \ldots, N\} \setminus \{h_1, \ldots, h_n\} \; .$$

$\Rightarrow$ Additive excitation spectrum.

$$\mathcal{P}_{\mathrm{ex}} - \mathcal{P}_{G.S.} = \sum_{a=1}^{n} p(\mu_{p_a}) - p(\mu_{h_a}) + \mathrm{O}(L^{-1}) \quad \text{and} \quad \mathcal{E}_{\mathrm{ex}} - \mathcal{E}_{G.S.} = \sum_{a=1}^{n} \varepsilon(\mu_{p_a}) - \varepsilon(\mu_{h_a}) + \mathrm{O}(L^{-1})$$

Excitations on the Fermi boundaries

⊛ n-particle hole excitations with macroscopic momenta $\{\mu_{p_a}\}_1^n$, $\{\mu_{h_a}\}_1^n$ *on* the Fermi surface

- n_h^+ holes and n_p^+ particles on right Fermi zone $\Rightarrow$ local deficiency $\ell \equiv n_p^+ - n_h^+$;
- n_h^- holes and n_p^- particles on left Fermi zone $\Rightarrow$ local deficiency $-\ell \equiv n_p^- - n_h^-$.

⇝ parametrization in terms of effective integers $h_a^\pm$ and $p_a^\pm$

$$\mu_{p_a} \sim q + \frac{2\pi}{L\rho(q)} p_a^+ \quad \text{or} \quad \mu_{p_a} \sim -q - \frac{2\pi}{L\rho(q)} p_a^-$$

$$\mu_{h_a} \sim q - \frac{2\pi}{L\rho(q)} h_a^+ \quad \text{or} \quad \mu_{h_a} \sim -q + \frac{2\pi}{L\rho(q)} h_a^-$$

- Simple form for the excitation momentum

$$\mathcal{P}_{\mathrm{ex}} - \mathcal{P}_{\mathrm{G.S.}} \sim 2\ell p_F + \frac{2\pi}{L}\left(\sum_{a=1}^{n_p^+} p_a^+ + \sum_{a=1}^{n_h^+} h_a^+\right) - \frac{2\pi}{L}\left(\sum_{a=1}^{n_p^-} p_a^- + \sum_{a=1}^{n_h^-} h_a^-\right).$$

Asymptotic behavior of form factors: the result

NLSE, **'90** Slavnov , *XX* **'06** Arikawa, Karbach, Müller, Wiele
6-Vertex R matrix **'09 - '10** Kitanine, Kozlowski, M., Slavnov, Terras

- excited state with particles $\mu_{p_1}, \ldots, \mu_{p_n}$ and holes $\mu_{h_1}, \ldots, \mu_{h_n}$.
- F shift function associated to such excitation.
- $\{\lambda_a\}_1^N$ GS distr. momenta, $\{\nu_a\}_1^{N'}$ excited state momenta.

Structure of form factors

Fermi repulsion-like behavior of the form factor (XXZ exact results : **'99** Kitanine, M., Terras)

$$\frac{\langle\, Excited \,|\mathcal{O}\,(0,0)\,|\, G.S.\,\rangle}{||Excited|| \cdot ||G.S.||} \sim \frac{\prod_{j<k}^{N}(\lambda_j - \lambda_k)\prod_{j>k}^{N'}(\nu_j - \nu_k)}{\prod_{k=1}^{N}\prod_{j=1}^{N'}(\lambda_k - \nu_j)} \times \underbrace{\mathscr{A}\begin{pmatrix} \mu_{p_1}, \ldots, \mu_{p_n} \\ \mu_{h_1}, \ldots, \mu_{h_n} \end{pmatrix}}_{\text{regular}} .$$

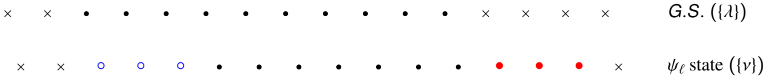
⊛ Extract the large volume L behavior $\Longrightarrow$ many cancellation of terms going to zero with L.

The power-law decay of form factors

⇝ Algebraic decay of form factors (in the volume L)

$$\left| \frac{\langle \, Excited \, | \mathcal{O}(0,0) | \, G.S. \, \rangle}{\|Excited\| \cdot \|G.S.\|} \right|^2 \sim \left(\frac{2\pi}{L} \right)^{\theta[F]} \cdot \underbrace{\mathcal{R}_n \begin{pmatrix} \{p_a\}; \{\mu_{p_a}\} \\ \{h_a\}; \{\mu_{h_a}\} \end{pmatrix} [F]}_{\text{discrete}} \cdot \underbrace{\mathcal{A}_n \begin{pmatrix} \{\mu_{p_a}\} \\ \{\mu_{h_a}\} \end{pmatrix}}_{\text{smooth}} .$$

⇝ Excitation on the Fermi boundary $\Longrightarrow$ description in terms of ℓ-shifted states



⊛ Local shifts of rapidities $N, L \gg s$:

$$\nu_{N-s} - \lambda_{N-s} \sim \frac{F_{\ell;+}}{L\rho(q)} \quad \text{right Fermi} \qquad \text{and} \qquad \nu_s - \lambda_s \sim \frac{F_{\ell;-}}{L\rho(-q)} \quad \text{left Fermi}$$

⊛ one value for volume power law behavior θ_ℓ.

Form factors of ℓ-shifted states

$$\left|\mathcal{F}_\ell\right|^2 = \lim_{L\to+\infty}\left\{L^{\theta_\ell}\left|\frac{\langle\, G.S.\,|\mathcal{O}|\,\psi_\ell\,\rangle}{\|G.S\|\cdot\|\psi_\ell\|}\right|^2\right\} \qquad \text{model/operator dependent}\;.$$

⊛ Form factors of any low-lying excitation with ℓ particles more on *right* Fermi zone:

$$\left|\frac{\langle\, Ex\,|\mathcal{O}\,(0,0)\,|\,G.S.\,\rangle}{\|Ex\|\;\cdot\;\|G.S.\|}\right|^2 \sim \frac{\left|\mathcal{F}_\ell\right|^2}{L^{\theta_\ell}} \times \frac{G^2(1+F_{\ell;+})G^2(1-F_{\ell;-})}{G^2(1+\ell+F_{\ell;+})G^2(1-\ell-F_{\ell;-})}\left(\frac{\sin(\pi F_{\ell;+})}{\pi}\right)^{2n_h^+}$$

$$\times\left(\frac{\sin(\pi F_{\ell;-})}{\pi}\right)^{2n_h^-} R_{n_p^+,n_h^+}\Big(\{p_a^+\},\{h_a^+\}\mid F_{\ell;+}\Big)R_{n_p^-,n_h^-}\Big(\{p_a^-\},\{h_a^-\}\mid -F_{\ell;-}\Big)\,.$$

⊛ **Red** part is universal. $G \rightsquigarrow$ Barnes function.

$$R_{n,m}\Big(\{p_a\}_1^n,\{h_a\}_1^m\mid F\Big) \equiv \frac{\prod_{j>k}^{n}(p_j-p_k)^2\prod_{j>k}^{m}(h_j-h_k)^2}{\prod_{j=1}^{n}\prod_{k=1}^{m}(p_j+h_k-1)^2}\prod_{k=1}^{n}\frac{\Gamma^2(p_k+F)}{\Gamma^2(p_k)}\prod_{k=1}^{m}\frac{\Gamma^2(h_k-F)}{\Gamma^2(h_k)}\,.$$

Form factor expansion of the generating function

$$\left\langle \mathcal{O}(x)\,\mathcal{O}^{\dagger}(0)\right\rangle = \sum_{\{\nu\}_{\mathrm{ex}}} \mathrm{e}^{ix(P_{\mathrm{G.S.}}-P_{\mathrm{ex}})} \left|\langle\, G.S.\,|\mathcal{O}(0,0)|\,\{\nu\}_{\mathrm{ex}}\,\rangle\right|^2$$

The $x \to +\infty$ asymptotics

- Only states having the same per-site energy as GS contribute in $L \to +\infty$;
- Only the individual leading in L behavior contributes to $L \to +\infty$ limit;

$$\left|\langle\, G.S.\,|\mathcal{O}(0,0)|\,\{\nu\}_{\mathrm{ex}}\,\rangle\right|^2 \sim L^{-\theta[\{\mu\}_{\mathrm{ex}}]}\,\mathcal{F}(\{\mu\}_{\mathrm{ex}})$$

$$\mathcal{P}_{\mathrm{ex}} - \mathcal{P}_{\mathrm{G.S.}} = \sum_{a=1}^{n} p(\mu_{p_a}) - p(\mu_{h_a}) + \mathrm{O}\left(L^{-1}\right) \qquad \mathcal{E}_{\mathrm{ex}} - \mathcal{E}_{\mathrm{G.S.}} = \sum_{a=1}^{n} \varepsilon(\mu_{p_a}) - \varepsilon(\mu_{h_a}) + \mathrm{O}\left(L^{-1}\right)$$

- Approximate summand at stationary points $\rightsquigarrow$ endpoints of Fermi zone,
- sum-up the resulting *critical* series.

The effective form of the series at $x \to +\infty$

$$\left\langle \mathcal{O}(x)\,\mathcal{O}^{\dagger}(0)\right\rangle \sim \lim_{N,L\to+\infty} \sum_{\ell\in\mathbb{Z}} e^{i2x\ell p_F} \cdot \left|\mathcal{F}_{\ell}\right|^{2} \cdot \mathscr{R}_{\ell}\big(x \mid F_{\ell;+}\big)\mathscr{R}_{-\ell}\big(-x \mid -F_{\ell;-}\big)$$

$$\mathscr{R}_{\ell}(x \mid \nu) = \left(\frac{2\pi}{L}\right)^{(\nu+\ell)^2} \frac{G^2(1+\nu)}{G^2(1+\nu+\ell)} \sum_{\substack{n_p,n_h\geq 0\\ n_p-n_h=\ell}} \sum_{\substack{p_1<\cdots<p_{n_p}\\ p_a\in\mathbb{N}^*}} \sum_{\substack{h_1<\cdots<h_{n_h}\\ h_a\in\mathbb{N}^*}} \left(\frac{\sin\pi\nu}{\pi}\right)^{2n_h} \prod_{a=1}^{n_p}\left\{e^{\frac{2i\pi}{L}p_a x}\right\} \cdot \prod_{a=1}^{n_h}\left\{e^{\frac{2i\pi}{L}(h_a-1)x}\right\}$$

$$\frac{\prod\limits_{a<b}^{n_p}(p_a-p_b)^2 \cdot \prod\limits_{a<b}^{n_h}(h_a-h_b)^2}{\prod\limits_{a=1}^{n_p}\prod\limits_{b=1}^{n_h}(p_a+h_b-1)^2} \cdot \prod_{a=1}^{n_p}\Gamma^2\left(\begin{array}{c} p_a+\nu \\ p_a \end{array}\right) \prod_{a=1}^{n_h}\Gamma^2\left(\begin{array}{c} h_a-\nu \\ h_a \end{array}\right)$$

The effective form of the series at $x \to +\infty$

$$\left\langle \mathcal{O}(x)\,\mathcal{O}^{\dagger}(0)\right\rangle \sim \lim_{N,L\to+\infty} \sum_{\ell\in\mathbb{Z}} \mathrm{e}^{i2x\ell p_F} \cdot \left|\mathcal{F}_\ell\right|^2 \cdot \mathscr{R}_\ell\big(x \mid F_{\ell;+}\big)\mathscr{R}_{-\ell}\big(-x \mid -F_{\ell;-}\big)$$

$$\mathscr{R}_\ell(x \mid \nu) = \left(\frac{2\pi}{L}\right)^{(\nu+\ell)^2} \frac{G^2(1+\nu)}{G^2(1+\nu+\ell)} \sum_{\substack{n_p,n_h\geq 0\\ n_p-n_h=\ell}} \sum_{\substack{p_1<\cdots<p_{n_p}\\ p_a\in\mathbb{N}^*}} \sum_{\substack{h_1<\cdots<h_{n_h}\\ h_a\in\mathbb{N}^*}} \left(\frac{\sin\pi\nu}{\pi}\right)^{2n_h} \prod_{a=1}^{n_p}\left\{\mathrm{e}^{\frac{2i\pi}{L}p_a x}\right\} . \prod_{a=1}^{n_h}\left\{\mathrm{e}^{\frac{2i\pi}{L}(h_a-1)x}\right\}$$

$$\frac{\prod_{a<b}^{n_p}(p_a-p_b)^2 \cdot \prod_{a<b}^{n_h}(h_a-h_b)^2}{\prod_{a=1}^{n_p}\prod_{b=1}^{n_h}(p_a+h_b-1)^2} \cdot \prod_{a=1}^{n_p}\Gamma^2\left(\begin{array}{c} p_a+\nu \\ p_a \end{array}\right) \prod_{a=1}^{n_h}\Gamma^2\left(\begin{array}{c} h_a-\nu \\ h_a \end{array}\right)$$

$$\mathscr{R}_\ell(x \mid \nu) = \left(\frac{2\pi/L}{1-\mathrm{e}^{\frac{2i\pi}{L}x}}\right)^{(\nu+\ell)^2}$$

- $\ell = 0$ Z-measures on partitions (**Kerov, Vershik, Borodin, Olshanski, Okounkov**) ;
- generalization to $\ell \neq 0$ and alternative proof at $\ell = 0$ (**'11, KKMST**).

The last step

$$\left\langle \mathcal{O}(x)\,\mathcal{O}^{\dagger}(0)\right\rangle \sim \lim_{N,L\to+\infty}\sum_{\ell\in\mathbb{Z}} \mathrm{e}^{i2x\ell p_F}\cdot\left|\mathcal{F}_{\ell}\right|^{2}\cdot\left(\frac{2\pi/L}{1-\mathrm{e}^{\frac{2i\pi}{L}x}}\right)^{(F_{\ell;+}+\ell)^{2}}\left(\frac{2\pi/L}{1-\mathrm{e}^{-\frac{2i\pi}{L}x}}\right)^{(F_{\ell;-}+\ell)^{2}} .$$

Now easy to send $L \to +\infty$

$$\left\langle \mathcal{O}(x)\,\mathcal{O}^{\dagger}(0)\right\rangle \sim \sum_{\ell\in\mathbb{Z}} \frac{\mathrm{e}^{i2x\ell p_F}\cdot\left|\mathcal{F}_{\ell}\right|^{2}}{(-ix)^{\Delta_{\ell;+}}\cdot(ix)^{\Delta_{\ell;-}}} .$$

Structure of the asymptotics

- Asymptotics indexed by Umklapp excitations ℓ
- $\left|\mathcal{F}_{\ell}\right|^{2}$ model dependent **but** universal interpretation
- Critical exponent $\Delta_{\ell;+}=(F_{\ell;+}+\ell)^{2}$ and $\Delta_{\ell;-}=(F_{\ell;-}+\ell)^{2}$

The n-point correlation functions

'14 Kitanine, Kozlowski, M., Terras

$$C\big(\boldsymbol{x}_r;\boldsymbol{o}_r\big) = \langle\, \Psi_g\, | \mathcal{O}_1(x_1)\ldots\mathcal{O}_r(x_r)|\, \Psi_g\,\rangle\,,$$

Local operators $\mathcal{O}_a(x)$ connect states with N and $N+o_a$ pseudo-particles; the form factor expansion given as a multiple sum over intermediate normalized states $|\,\Psi\big(\mathcal{I}_n^{(s)}\big)\,\rangle$ with $s=1,\ldots r-1$, labelled by sets of integers corresponding to particles and holes excitations :

$$\mathcal{I}_n^{(s)} = \Big\{ \{p_a^{(s)}\}_1^n \quad ; \quad \{h_a^{(s)}\}_1^n \Big\}$$

$$\langle\, \Psi\big(\mathcal{I}_m^{(s-1)}\big)\,|\mathcal{O}_s(x)|\,\Psi\big(\mathcal{I}_n^{(s)}\big)\,\rangle = \mathrm{e}^{ix(\Delta\mathcal{P})_{s-1}^{s}} \cdot \mathcal{F}_{\mathcal{O}_s}\big(\mathcal{I}_m^{(s-1)}\mid\mathcal{I}_n^{(s)}\big)$$

$$(\Delta\mathcal{P})_{s-1}^{s} = \mathcal{P}_{\mathcal{I}_m^{(s-1)}} - \mathcal{P}_{\mathcal{I}_n^{(s)}}$$

$$C\big(\boldsymbol{x}_r;\boldsymbol{o}_r\big) = \prod_{s=1}^{r-1}\Big\{ \sum_{\{\mathcal{I}_{n^{(s)}}^{(s)}\}} \Big\} \cdot \prod_{s=1}^{r-1}\Big\{ \exp\Big[i(x_{s+1}-x_s)\cdot\Delta\mathcal{P}(\mathcal{I}_{n^{(s)}}^{(s)})\Big]\Big\} \cdot \prod_{s=1}^{r} \mathcal{F}_{\mathcal{O}_s}\Big(\mathcal{I}_{n^{(s-1)}}^{(s-1)} \mid \mathcal{I}_{n^{(s)}}^{(s)}\Big)$$

General form factors (1)

$$\mathcal{F}_{O_s}\left(\mathcal{I}_m^{(s-1)}\middle|\mathcal{I}_n^{(s)}\right) = \mathcal{F}_{O_s}(\ell_{s-1},\ell_s)\cdot C^{(\ell_{s-1};\ell_s)}\left(\nu_s^+,\nu_s^-\right)\times \mathscr{F}^{(+)}\left[\mathcal{J}^{(s-1)}_{m_{p;+};m_{h;+}};\mathcal{J}^{(s)}_{n_{p;+};n_{h;+}} \mid \nu_s^+\right]\cdot$$

$$\cdot\mathscr{F}^{(-)}\left[\mathcal{J}^{(s-1)}_{m_{p;-};m_{h;-}};\mathcal{J}^{(s)}_{n_{p;-};n_{h;-}} \mid \nu_s^-\right]$$

$$\mathscr{F}_{O_s}(\ell_{s-1},\ell_s) = \lim_{L\to+\infty}\left\{\left(\frac{L}{2\pi}\right)^{\rho_s(\nu_s^+)+\rho_s(\nu_s^-)}\langle\,\Psi\left(\mathcal{L}^{(s-1)}_{\ell_{s-1}}\right)|O_s(0)|\,\Psi\left(\mathcal{L}^{(s)}_{\ell_s}\right)\rangle\right\}$$

$$\rho_s(\nu) = \frac{1}{2}(\ell_s-\ell_{s-1})^2 + \frac{1}{2}\nu^2 - (\ell_s-\ell_{s-1})\nu\,.$$

$$\nu_s^+ = \nu_s(q) - o_s \qquad \text{and} \qquad \nu_s^- = \nu_s(-q)$$

in terms of the relative shift function between the ℓ_s,ℓ_{s-1} critical states

$$\nu_s(\lambda) = F_{s-1}(\lambda) - F_s(\lambda)\,.$$

General form factors (2)

The right Fermi boundary critical form factor reads :

$$\mathscr{F}^{(+)}\Big[\mathcal{J}_{n_p;n_h};\mathcal{J}_{n_k;n_t}\mid\nu\Big] = \left(\frac{2\pi}{L}\right)^{\rho_s(\nu)}(-1)^{n_t}\left(\frac{\sin[\pi\nu]}{\pi}\right)^{n_t+n_h}\varpi\Big(\mathcal{J}_{n_p;n_h};\mathcal{J}_{n_k;n_t}\mid\nu\Big)\cdot$$

$$\frac{\prod\limits_{a<b}^{n_p}(p_a-p_b)\prod\limits_{a<b}^{n_h}(h_a-h_b)}{\prod\limits_{a=1}^{n_p}\prod\limits_{b=1}^{n_h}(p_a+h_b-1)}\frac{\prod\limits_{a<b}^{n_k}(k_a-k_b)\prod\limits_{a<b}^{n_t}(t_a-t_b)}{\prod\limits_{a=1}^{n_k}\prod\limits_{b=1}^{n_t}(k_a+t_b-1)}\Gamma\left(\begin{matrix}\{p_a+\nu\} & \{h_a-\nu\} & \{k_a-\nu\} & \{t_a+\nu\}\\ \{p_a\} & \{h_a\} & \{k_a\} & \{t_a\}\end{matrix}\right)$$

$$\varpi\Big(\mathcal{J}_{n_p;n_h};\mathcal{J}_{n_k;n_t}\mid\nu\Big) = \prod_{a=1}^{n_h}\left\{\frac{\prod\limits_{b=1}^{n_k}\Big(1-k_b-h_a+\nu\Big)}{\prod\limits_{b=1}^{n_t}\Big(t_b-h_a+\nu\Big)}\right\}\cdot\prod_{a=1}^{n_p}\left\{\frac{\prod\limits_{b=1}^{n_t}\Big(p_a+t_b+\nu-1\Big)}{\prod\limits_{b=1}^{n_k}\Big(p_a-k_b+\nu\Big)}\right\}.$$

This ϖ term couples the right and left states particles and holes integers (not present if one of them is the ground state) hence leading to coupling of previous combinatorial sums!

General sums (1)

$$C\big(\boldsymbol{x}_r;\boldsymbol{o}_r\big) \simeq \sum_{\substack{\boldsymbol{\ell}_{r-1}\\ \in\mathbb{Z}^{r-1}}} \Big(\frac{2\pi}{L}\Big)^{\vartheta(\boldsymbol{\ell}_{r-1},\boldsymbol{o}_r)} \prod_{s=1}^{r-1}\Big\{\mathrm{e}^{2i\ell_s(x_{s+1}-x_s)p_F}\Big\} \prod_{s=1}^{r}\Big\{C^{(\ell_{s-1};\ell_s)}\big(\nu_s^+,\nu_s^-\big)\Big\}\cdot$$

$$\prod_{s=1}^{r}\big\{\mathcal{F}_{O_s}(\ell_{s-1},\ell_s)\big\}\,\mathscr{S}^-_{\boldsymbol{\ell}_{r-1}}\Big(\Big\{\frac{2\pi}{L}(x_{s+1}-x_s)\Big\}_1^{r-1},\{\nu_s^-(\boldsymbol{\ell}_s)\}_1^r\Big)\,\mathscr{S}^+_{\boldsymbol{\ell}_{r-1}}\Big(\Big\{\frac{2\pi}{L}(x_{s+1}-x_s)\Big\}_1^{r-1},\big\{\nu_s^+(\boldsymbol{\ell}_s)\big\}_1^r\Big)$$

$$\vartheta(\boldsymbol{\ell}_{r-1},\boldsymbol{o}_r) = \frac{1}{2}\sum_{s=1}^{r}\Big\{(\nu_s^+)^2+(\nu_s^-)^2\Big\} - \sum_{s=1}^{r-1}\Big\{\big(\nu_s^+ +\nu_s^- -\nu_{s+1}^+ -\nu_{s+1}^-\big)\ell_s - 2\ell_s^2\Big\} - 2\sum_{s=2}^{r-1}\ell_s\ell_{s-1}$$

$$\mathscr{S}^\pm_{\boldsymbol{\ell}_{r-1}}(\{t_s\},\{\nu_s\}) = \prod_{s=1}^{r-1}\sum_{\substack{n_p^{(s)},n_h^{(s)}=0\\ n_p^{(s)}-n_h^{(s)}=\pm\ell_s}}^{+\infty}\sum_{\mathcal{J}^{(s)}_{n_p^{(s)};n_h^{(s)}}}\prod_{s=1}^{r-1}\mathcal{R}^\pm(\mathcal{J}^{(s)}_{n_p^{(s)};n_h^{(s)}}|\nu_s,\nu_{s+1};t_s)\prod_{s=2}^{r-1}\varpi\big(\mathcal{J}^{(s-1)}_{n_p^{(s-1)};n_h^{(s-1)}};\mathcal{J}^{(s)}_{n_p^{(s)};n_h^{(s)}}\mid\pm\nu_s\big)$$

The summation in the above formula runs through all the possible choices of the sets of integers that parametrize the states

$$\mathcal{J}^{(s)}_{n_p^{(s)};n_h^{(s)}} = \Big\{\{p_a^{(s)}\}_1^{n_p^{(s)}}\ ;\ \{h_a^{(s)}\}_1^{n_h^{(s)}}\Big\}$$

General sums (2)

Amazingly, these generalized combinatorial sums can be computed exactly!

$$\mathscr{S}^{\pm}_{\boldsymbol{\ell}_{r-1}}\left(\{t_s\}_1^{r-1},\{\nu_s\}_1^r\right) = \prod_{s=1}^{r-1}\left\{ e^{\pm i t_s \frac{\ell_s(\ell_s+1)}{2}} G\left(\begin{array}{c} 1\pm(\ell_s-\nu_s), 1\pm(\ell_s+\nu_{s+1}) \\ 1\mp\nu_s, 1\pm\nu_{s+1} \end{array} \right)\right\}$$

$$\times \prod_{s=2}^{r-1} G\left(\begin{array}{c} 1\pm\nu_s, 1\pm(\ell_{s-1}-\ell_s+\nu_s) \\ 1\mp(\ell_s-\nu_s), 1\pm(\ell_{s-1}+\nu_s) \end{array}\right) \cdot \prod_{b>a}^{r}\left(1-e^{\pm i \sum_{s=a}^{b-1} t_a}\right)^{(\nu_a+\kappa_a)(\nu_b+\kappa_b)}$$

$$\kappa_s = \ell_{s-1}-\ell_s \quad \text{for} \quad s=1,\ldots,r \qquad \text{so that} \qquad \sum_{a=1}^{r}\kappa_a = 0.$$

$$C\left(\boldsymbol{x}_r;\boldsymbol{o}_r\right) = \sum_{\substack{\boldsymbol{\kappa}_r\in\mathbb{Z}^r \\ \sum\kappa_a=0}} \prod_{s=1}^{r}\{e^{2ip_F\kappa_s x_s}\}\cdot\mathcal{F}\left(\{\kappa_a\}_1^r;\{o_a\}_1^r\right)\cdot$$

$$\prod_{s=1}^{r}\left(\frac{2\pi}{L}\right)^{\frac{1}{2}[\theta_s^+(\kappa_s)]^2+\frac{1}{2}[\theta_s^-(\kappa_s)]^2}\prod_{b>a}^{r}\left\{\left[1-e^{\frac{2i\pi}{L}(x_b-x_a)}\right]^{\theta_b^+(\kappa_b)\theta_a^+(\kappa_a)}\cdot\left[1-e^{-\frac{2i\pi}{L}(x_b-x_a)}\right]^{\theta_b^-(\kappa_b)\theta_a^-(\kappa_a)}\right\}$$

$$\theta_b^{\pm}(\kappa_b) = \nu_b^{\pm}+\kappa_b$$

Asymptotic behavior of n-point correlation functions

Taking the thermodynamic limit we arrive at the following n-point correlation function asymptotic behavior :

$$C\Big(\boldsymbol{x}_r;\boldsymbol{o}_r\Big) = \sum_{\substack{\boldsymbol{\kappa}_r\in\mathbb{Z}^r \\ \sum\kappa_a=0}} \prod_{s=1}^{r}\Big\{\mathrm{e}^{2ip_F\kappa_s x_s}\Big\}\cdot\mathcal{F}\Big(\{\kappa_a\}_1^r;\{o_a\}_1^r\Big)$$

$$\prod_{b>a}^{r}\left\{\Big[\,\mathrm{i}(x_b-x_a)\Big]^{\theta_b^-(\kappa_b)\theta_a^-(\kappa_a)}\cdot\Big[-i(x_b-x_a)\Big]^{\theta_b^+(\kappa_b)\theta_a^+(\kappa_a)}\right\}.$$

Note that the above asymptotic expansion provides one with an expression that is symmetric under a simultaneous permutation

$$\Big(\boldsymbol{x}_r,\boldsymbol{o}_r\Big)\mapsto\Big(\boldsymbol{x}_r^\sigma,\boldsymbol{o}_r^\sigma\Big)\quad\text{with}\quad\boldsymbol{x}_r^\sigma=\Big(x_{\sigma(1)},\dots,x_{\sigma(r)}\Big)\qquad\sigma\in\mathfrak{S}_r\,.$$

This is directly related to locality, namely to the fact that the local operators $\mathcal{O}_r(x_r)$ commute at different distances and, in particular, in the long-distance regime.

Example : four point function of XXZ

Consider the four point function:

$$C_{xxxx} = \langle \Psi_g | \sigma^x_{m_1} \, \sigma^x_{m_2} \, \sigma^x_{m_3} \, \sigma^x_{m_4} | \Psi_g \rangle.$$

The leading term confirms the Luther-Peschel prediction:

$$C_{xxxx} = 2 \left| \mathcal{F}_0^+ \right|^4 \cdot \left\{ \left| \frac{(m_2 - m_1) \cdot (m_4 - m_3)}{(m_3 - m_1) \cdot (m_4 - m_1) \cdot (m_3 - m_2) \cdot (m_4 - m_2)} \right|^{\frac{1}{2\mathcal{Z}^2}} + (2 \leftrightarrow 3) + (2 \leftrightarrow 4) \right\} + \dots .$$

$\hookrightarrow$ It gives the possibility to analyze de product structure of the amplitudes in terms of elementary form factors of local operators.

Conclusion and perspectives

Results

- ✔ Leading asymptotics of **any** harmonic in long-distance
- ✔ **All** harmonics in long-distance and large-time for pure particle-hole spectrum
- ✔ Reproduction of edge exponents with amplitudes from ABA
- ✔ Temperature case (Kozlowski, M. , Slavnov and Dugave, Göhmann, Kozlowski)
- ✔ Leading asymptotic behavior of n-point correlation functions

What's next?

- ⊛ Include the effects of bound states (time dependent case)
- ⊛ Full link with CFT (OPE of local operators + structure constants)