# Large-N asymptotic expansion of multiple integrals related to the quantum separation of variables method

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# Outline

# The quantum separation of variables

⊛ ideas go back to **'79** **Kostant** , **'82-'86** **Goodmann,Wallach** , **'80** **Gutzwiller**

⇝ Circumvent certain limitations of ABA (**'85** **Sklyanin** )

$$\mathcal{U}^{-1} \ : \ \mathfrak{h}_{\mathrm{org}} = \mathfrak{h}_1 \otimes \cdots \otimes \mathfrak{h}_N \ \rightarrow \ L^2\big(\mathbb{R}^N, \mathrm{d}\mu_N(y)\big)$$

$$\Phi(\vec{x}) = \int_{\mathbb{R}^N} \mathcal{U}(\vec{x}, \vec{y}) \cdot \widehat{\Phi}(\vec{y}) \cdot \mathrm{d}\mu_N(\vec{y}) \ .$$

$\mathcal{U}$ : Multi-parameter multi-dimensional SP $\mapsto$ Multi-parameter *one*-dimensional SP

$$\widehat{\Phi}_{\mathrm{ev}}(\vec{y}) \quad \hookrightarrow \quad \prod_{a=1}^{N} Q(y_a) \quad \text{with} \quad Q(y) = \mathrm{e}^{-\frac{1}{2}V(y)}$$

- many models (Lattice Sinh & Sine-G, anti-periodic XXZ, ...);
- Builds on a "universal" structure of the space of states;
- Very efficient for describing spectrum: $T-Q$ or NLIE;
- $1+1$ dim QFT's in finite volume through $N \rightarrow +\infty$.

✔ Spectrum at $N \rightarrow +\infty$ through NLIE methods (**'90** **Batchelor,Klumper** ;**'92** **Destri,De Vega** )

# The quantum separation of variables : correlators

**What about correlation functions in the continuum?**

⊛ Norms in the quantum separation of variables (*eg*. Lattice Sinh-G):

$$\int_{\mathbb{R}^N} \left|\Phi(\vec{x})\right|^2 \cdot \mathrm{d}^N x = \int_{\mathbb{R}^N} \prod_{a<b}^{N} \Big\{ \sinh[\pi\omega_1(y_a - y_b)] \cdot \sinh[\pi\omega_2(y_a - y_b)] \Big\} \prod_{a=1}^{N} \mathrm{e}^{-V(y_a)} \cdot \mathrm{d}^N y$$

♦ Expressions on $L^2(\mathbb{R}^N, \mathrm{d}\mu_N(y))$ for certain local operators exist ('04 **Oota** , '04 **Babelon** , '12 **Grosjean, Maillet, Niccoli** ,'13 **Sklyanin** , '13 **K.** )

⇝ Similar multiple integral expressions as norms.

**Extract the $N \rightarrow +\infty$ of the multiple integral to describe QFT correlators**

**Main message**

- adaptation of ideas used in random matrix theory is possible
- new approach to loop equations and new class of special functions had to be build
- the large-$N$ asymptotics can be explicited in various cases.

# Examples of leading large-$N$ asymptotics

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- Multiple integral of interest

$$\mathfrak{z}_N = \int_{\mathbb{R}^N} \prod_{a<b}^{N} \Big\{ \sinh[\pi\omega_1(y_a - y_b)] \cdot \sinh[\pi\omega_2(y_a - y_b)] \Big\}^{\beta} \cdot \prod_{a=1}^{N} \mathrm{e}^{-V(y_a)} \cdot \mathrm{d}^N y$$

  - $\omega_1, \omega_2, \beta > 0$ free parameters;
  - $V$ is confining $\lim_{y\to\pm\infty} \frac{V(y)}{|y|^{1+\epsilon}} = +\infty$

**Leading asymptotics are driven by the asymptotics of $V$**

- Polynomial asymptotics $V(y) \sim c_q \cdot |y|^q$ for $y \to \pm\infty$

$$\ln \mathfrak{z}_N = N^{2+\frac{1}{q-1}} \cdot \Big( \Big[ \frac{\pi\beta}{q}(\omega_1 + \omega_2) \Big]^{\frac{q+1}{q}} c_q^{\frac{1}{q}} \frac{2q^2-9q+6}{2(2q-1)} + \mathrm{o}(1) \Big)$$

- Hyperbolic asymptotics $V(y) \sim R\cosh(y)$ for $y \to \pm\infty$

$$\ln \mathfrak{z}_N = N^2 \ln N \cdot \Big( 2\pi\beta(\omega_1 + \omega_2) + \mathrm{o}(1) \Big)$$

- subleading asymptotics demand much more care:
  ⇝ calculations build on a rescaled model ;

# $\beta$-ensemble integrals with varying weights

$$\mathcal{Z}_N^{(\beta)} = \int\limits_{\mathbb{R}^N} \prod_{a<b}^{N} |\lambda_a - \lambda_b|^{\beta} \cdot \prod_{a=1}^{N} \mathrm{e}^{-NV(\lambda_a)} \cdot \mathrm{d}^N\lambda$$

- Long activity to establish the large-$N$ expansion :

  '78 **Brézin, Itzykson, Parisi, Zuber** , '93 **Ambjorn, Chekhov,Kristjansen, Makeenko** , '95 **Boutet de Monvel, Pastur, Shcherbina** ,'97 **Ben Arous, Guionnet** , '98 **Johansson** , '04 **Eynard** , '13 **Borot, Guionnet**

  Intgerand represented as $\exp\{-N^2\mathcal{E}^{(\beta)}[L_N]\}$ with

$$\mathcal{E}^{(\beta)}[\mu] = \int V(\xi)\mathrm{d}\mu(\xi) - \frac{\beta}{2}\int\limits_{\xi\neq\eta} \ln|\xi-\eta| \cdot \mathrm{d}\mu(\xi)\otimes\mathrm{d}\mu(\eta) \qquad \text{and} \qquad L_N = \frac{1}{N}\sum_{a=1}^{N}\delta_{\lambda_a}$$

**Leading asymptotics**: $\ln \mathcal{Z}_N^{(\beta)} = -N^2\mathcal{E}^{(\beta)}[\mu_{\mathrm{eq}}] + \mathrm{O}(N)$

- $\mu_{\mathrm{eq}} \equiv$ unique minimiser on $\mathcal{M}^1(\mathbb{R})$ of $\mathcal{E}^{(\beta)}$;
- density $\rho_{\mathrm{eq}}(\xi) = \mathrm{d}\mu_{\mathrm{eq}}(\xi)/\mathrm{d}\xi$ solves *scalar* RHP $\rightsquigarrow$ one-fold integral representation;
- $L_N \hookrightarrow \mu_{\mathrm{eq}}$ : variables condensate on $\mathrm{supp}[\mu_{\mathrm{eq}}]$ with a density $\rho_{\mathrm{eq}}$ ;
- $\mathrm{O}(N) \equiv$ contribution of fluctuations around the saddle-point configuration.

# All orders asymptotic expansion of $\beta$-ensembles with varying weights

♦ *Topological recursion* of loop equations for "correlators" ( =*functions* )

↪ all order asymptotic expansion for real analytic convex $V$:

$$\ln \mathcal{Z}_N^{(\beta)} \simeq \sum_{k=0}^{+\infty} N^{2-k} c_k[V] \qquad \text{with} \qquad c_0[V] = -\mathcal{E}^{(\beta)}[\mu_{\mathrm{eq}}]$$

- Only one scale $N^{-1}$ drives the asymptotics;
- $c_k[V]$ functionals of $V$, explicitly computable order by order;
- slightly more complex expression for real analytic $V$.

# $\beta$-ensembles with *non*-varying weights

♦ Things become problematic when $V$ is *not* varying with $N$ (**'00 Zinn-Justin**, **'06 Bleher, Fokin**):

$$\mathcal{Z}_N^{(\beta)} = \int_{\mathbb{R}^N} \underbrace{\prod_{a<b}^{N} |\lambda_a - \lambda_b|^{\beta}}_{e^{O(N^2)}} \cdot \underbrace{\prod_{a=1}^{N} e^{-V(\lambda_a)}}_{e^{O(N)}} \cdot \underbrace{d^N\lambda}_{e^{O(N)}} \qquad V(\xi) = \frac{\xi^4}{1+\xi^2} \underset{\xi\to\pm\infty}{\sim} \xi^2$$

$\Longrightarrow$ The $\beta$-interaction part dominantes on bounded subsets of $\mathbb{R}^N$:
$\hookrightarrow$ $\lambda_a$ spread ... until $V(\xi)$ kicks in

Rescale variables $\lambda_a = T_N \mu_a$

$$\mathcal{Z}_N^{(\beta)} = \gamma_N \int_{\mathbb{R}^N} \underbrace{\prod_{a<b}^{N} |\mu_a - \mu_b|^{\beta}}_{e^{O(N^2)}} \cdot \underbrace{\prod_{a=1}^{N} e^{-T_N^2 V_N(\mu_a)}}_{e^{O(T_N^2 N)}} \cdot \underbrace{d^N\mu}_{e^{O(N)}} \qquad V_N(\xi) = \frac{\xi^4 T_N^2}{1+T_N^2\xi^2} = \xi^2 + \cdots$$

- Equality between scales for $T_N^2 = N$ ;
- Trivial scaling pre-factor from $\beta$-interactions $\gamma_N = \big(T_N\big)^{N+\frac{\beta}{2}N(N-1)}$;
- Singularities of $V_N$ squeeze down to $\mathbb{R}$ with $N$ .

# The scaled integral

- Potential with polynomial asymptotics $V(\xi) \underset{\xi\to\pm\infty}{\sim} |\xi|^q$.

$$\mathfrak{z}_N = \int_{\mathbb{R}^N} \underbrace{\prod_{a<b}^{N} \big\{\sinh[\pi\omega_1(y_a - y_b)] \cdot \sinh[\pi\omega_2(y_a - y_b)]\big\}^{\beta}}_{e^{O(N^2)}} \cdot \underbrace{\prod_{a=1}^{N} e^{-|y_a|^q}}_{e^{O(N)}} \cdot \underbrace{d^N y}_{e^{O(N)}}$$

$\Longrightarrow$ The sinh-part dominantes on bounded subsets of $\mathbb{R}^N$:

$\hookrightarrow$ $y_a$ spread ... until $V$ kicks in

Rescale variables $y_a = T_N \lambda_a$

$$\mathfrak{z}_N = \big(T_N\big)^N \int_{\mathbb{R}^N} \underbrace{\prod_{a<b}^{N} \big\{\sinh[\pi T_N\omega_1(\lambda_a - \lambda_b)] \cdot \sinh[\pi T_N\omega_2(\lambda_a - \lambda_b)]\big\}^{\beta}}_{e^{O(N^2 T_N)}} \cdot \underbrace{\prod_{a=1}^{N} e^{-T_N^q V_N(\lambda_a)}}_{e^{O(N T_N^q)}} \cdot \underbrace{d^N \lambda}_{e^{O(N)}}$$

Equilibruim of orders of magnitude $N^2 T_N = T_N^q N$ *ie.* $T_N = N^{\frac{1}{q-1}}$

- $N$-dependent potential $V_N(\xi) = N^{-\frac{q}{q-1}} V_N\Big(N^{\frac{1}{q-1}}\xi\Big)$ ;
- two-body interactions scale with $N$.

# The large-$N$ asymptotic expansion of the rescaled integral

$$\mathcal{Z}_N[V] = \int_{\mathbb{R}^N} \prod_{a<b}^{N} \Big\{ \sinh[\pi N^{\alpha}\omega_1(\lambda_a-\lambda_b)] \cdot \sinh[\pi N^{\alpha}\omega_2(\lambda_a-\lambda_b)] \Big\}^{\beta} \cdot \prod_{a=1}^{N} \mathrm{e}^{-N^{1+\alpha}V(\lambda_a)} \cdot \mathrm{d}^N\lambda$$

**Hypothesis**

- $V$ is strictly convex and $C^k(\mathbb{R})$, $k \geq 2$;
- $0 \leq \alpha \leq 1/6$.

⊛ **Step I** Reabsorbtion of non-topological contributions to all orders in $N^{-\alpha}$

$$\mathcal{Z}_N[V] = -N^{2+\alpha} \cdot \underbrace{\mathcal{E}_N\Big[\mu_{\mathrm{eq}}^{(N)}\Big]}_{\text{non-topological}} + \underbrace{\mathrm{O}\Big(N^{1+\alpha}\Big)}_{\text{Lebesgue}} \qquad \text{with} \qquad \mathcal{E}_N\Big[\mu_{\mathrm{eq}}^{(N)}\Big] = \inf\Big\{\mathcal{E}_N[\nu] \; : \; \nu \in \mathcal{M}^1(\mathbb{R})\Big\}$$

by using $N$-dependent rate function

$$\mathcal{E}_N[\nu] = \int V(\xi)\mathrm{d}\nu(\xi) - \frac{\beta}{2N^{\alpha}} \int \ln\Big[\sinh[\pi N^{\alpha}\omega_1(\xi-\eta)] \cdot \sinh[\pi N^{\alpha}\omega_2(\xi-\eta)]\Big] \mathrm{d}\nu(\xi) \otimes \mathrm{d}\nu(\eta) \, .$$

# The equilibrium measure

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- Unique minimiser of $\mathcal{E}_N$ ;
- supported on $[\, a_N \, ; b_N \,]$ and Lebesgue continuous $\mathrm{d}\mu_{\mathrm{eq}}^{(N)}(\xi) = \rho_{\mathrm{eq}}^{(N)}(\xi)\mathrm{d}\xi$ ;
- $\rho_{\mathrm{eq}}^{(N)}(\xi) \sim C_{N;\varsigma}|\xi-\varsigma|^{\frac{1}{2}}$ when $\varsigma \to a_N, b_N$ & $\int_{a_N}^{b_N} \rho_{\mathrm{eq}}^{(N)}(\xi)\cdot\mathrm{d}\xi = 1$ ;
- $\rho_{\mathrm{eq}}^{(N)}$ solves a truncated Wiener–Hopf equation ;

$$V'(\xi) = \mathcal{S}_N\big[\rho_{\mathrm{eq}}^{(N)}\big] = \fint_{a_N}^{b_N} S\big(N^{\alpha}(\xi-\eta)\big)\rho_{\mathrm{eq}}^{(N)}(\eta)\cdot\mathrm{d}\eta \qquad \text{with} \qquad S(x) = \beta\pi\sum_{a=1}^{2}\omega_a \coth[\pi\omega_a x]$$

♦ $\mathcal{S}_N$ invertible on $H_s(\mathbb{R})$, $s<0$, when $N \geq N_0$, for all $a_N, b_N$ through solution of $2\times 2$ RHP
$\rightsquigarrow$ in spirit of the class of singular operators treated by Novoksenov **'80**

- Solve the RHP for $N$-large;
- characterise $\mathcal{S}_N[H_{1/2}(\mathbb{R})]$ and describe $\mathcal{S}_N^{-1} \mid \mathcal{S}_N[H_{1/2}(\mathbb{R})]$

# An integral representation for $\rho_{\mathrm{eq}}^{(N)}$

⊛ Integral representation for the inverse

$$\rho_{\mathrm{eq}}^{(N)}(\xi) = \mathcal{W}_N\big[V'\big](\xi) \; = \int_{a_N}^{b_N} \underbrace{W_N(\xi,\eta)}_{2\times 2\ RHP} V'(\eta)\cdot \mathrm{d}\eta$$

⊛ Constraints fix $a_N$ and $b_N$: $V'(b) = -V'(a) = \pi\beta(\omega_1+\omega_2)$

$$a_N \sim a + \sum_{k\geq 1} \frac{a_k}{N^{k\alpha}} \qquad \text{and} \qquad b_N \sim b + \sum_{k\geq 1} \frac{b_k}{N^{k\alpha}}$$

⊛ Develop the local and asymptotic theory for the new class of functions

- Uniform in $\xi, N$ local behaviour (bulk and boundaries);
- factorisation into smooth and singular (sqaure-root like) parts.

# A two-scaled asymptotic expansion

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**Theorem** Let $V$ be strictly convex and $0 \leq \alpha < 1/6$

$$\ln \mathcal{Z}_N[V] \;=\; N^{2+\alpha} \sum_{k,\ell \geq 0} \frac{c_{k,\ell}[V]}{N^k N^{\ell\alpha}} \;+\; \mathrm{o}(1)$$

- $c_{k,\ell}[V]$ are **explicit** distributions with support $[\,a\,;\,b\,]$.
- Leading term

$$c_{0,0}[V] \;=\; \frac{-1}{4\pi\beta(\omega_1+\omega_2)} \Big\{ \big(V'(b)\big)^2 \cdot (b-a) \;+\; \int_a^b \big(V'(\xi)\big)^2 \cdot \mathrm{d}\xi \Big\}$$

- Constant term

$$c_{2,1}[V] \;=\; \aleph \cdot \Big( \mho'[V](b) \;+\; \mho'[V](a) \Big)$$

with $$\mho[V,V_0](\xi) \;=\; \frac{V'(\xi)-\kappa\xi-\tau}{V''(\xi)-\kappa} \;+\; \frac{(\kappa\xi+\tau)V''(\xi)-V'(\xi)\kappa}{\big(V''(\xi)-\kappa\big)^2} \cdot \ln\Big(\frac{V''(\xi)}{\kappa}\Big)$$

$$\kappa \;=\; 2\pi\beta \frac{\omega_1+\omega_2}{b-a} \qquad \text{and} \qquad \tau \;=\; 2\pi\beta(\omega_1+\omega_2)\frac{b+a}{b-a}$$

# The constant $\aleph$

$$\aleph = -\frac{1}{2}\int_{\mathbb{R}} \frac{\mathrm{d}u}{\pi} J(u) \int_{|u|}^{+\infty} \mathrm{d}v \frac{\partial}{\partial u}\Big\{ S(u)\Big[\mathfrak{r}\big(\tfrac{v-u}{2}\big) - \mathfrak{r}\big(\tfrac{v+u}{2}\big)\Big]\Big\} +$$

$$\int_{\mathscr{C}} \frac{\mathrm{d}\lambda}{2i\pi} \int_{\mathscr{C}} \frac{\mathrm{d}\lambda}{2i\pi} \frac{\mu}{(\lambda+\mu)\cdot\alpha(\lambda)\cdot\alpha(\mu)} \int_{0}^{+\infty} \mathrm{d}x\mathrm{d}y \mathrm{e}^{ix\lambda+iy\mu} \frac{\partial}{\partial x}\Big\{ S(x-y)\cdot\big(\mathfrak{r}(x)-\mathfrak{r}(y)\big)\Big\}$$

- The function $J$

$$J(x) = \frac{\partial}{\partial x} \ln\Bigg[ \frac{\Big|1+\mathrm{e}^{-2\pi x\frac{\omega_1\omega_2}{\omega_1+\omega_2}+i\pi\frac{\omega_2-\omega_1}{\omega_1+\omega_2}}\Big|}{1-\mathrm{e}^{-2\pi x\frac{\omega_1\omega_2}{\omega_1+\omega_2}}} \Bigg]$$

- $\alpha \rightsquigarrow$ Wiener–Hopf factor of $1/\mathcal{F}[S]$:

$$\alpha(\lambda) = \frac{\lambda}{2\pi\sqrt{\omega_1+\omega_2}} \cdot \Big(\frac{\omega_2}{\omega_1+\omega_2}\Big)^{-\frac{i\lambda}{2\pi\omega_1}} \cdot \Big(\frac{\omega_1}{\omega_1+\omega_2}\Big)^{-\frac{i\lambda}{2\pi\omega_2}} \cdot \frac{\Gamma\Big(\frac{i\lambda}{2\pi\omega_1}\Big)\Gamma\Big(\frac{i\lambda}{2\pi\omega_1}\Big)}{\Gamma\Big(\frac{i\lambda(\omega_1+\omega_2)}{2\pi\omega_1\omega_2}\Big)}$$

- $\mathfrak{r}$ built through $\alpha$

$$\mathfrak{r}(x) = \sum_{a=1}^{2} \frac{\omega_1+\omega_2}{\pi\omega_a} \ln\Big(\frac{\omega_1\omega_2}{\omega_a(\omega_1+\omega_2)}\Big) \cdot \frac{\mathfrak{b}(x)}{1+2\pi\beta(\omega_1+\omega_2)\mathfrak{b}(x)}$$

$$\mathfrak{b}(x) = \frac{-i}{2\pi\beta\sqrt{\omega_1+\omega_2}} \int_{\mathscr{C}} \frac{\mathrm{e}^{i\lambda x}}{\lambda\alpha(\lambda)} \frac{\mathrm{d}\lambda}{2i\pi}$$

# Conslusion and perspectives

## Review of the results

- ✔ Method for analysis of multiple integrals with *non*-varying interactions ;
- ✔ Asymptotic expansion in "non-critical" cases ;
- ✔ Complete characterisation of equilibrium measure and master operator by $2 \times 2$ RHP ;
- ✔ Method allows to treat $\beta$-ensembles with $C^k$ potential ;
- ✔ Works for any 2-body interaction $G(\lambda, \mu) = \widetilde{G}(\lambda - \mu)$.

## Further developments

- ⊛ Treat the case of "critical" potentials as *eg.* in lattice sinh-G model .
- ⊛ Extend the method to less "explicit" interactions.