Level Two String Functions and Rogers Ramanujan Type Identities

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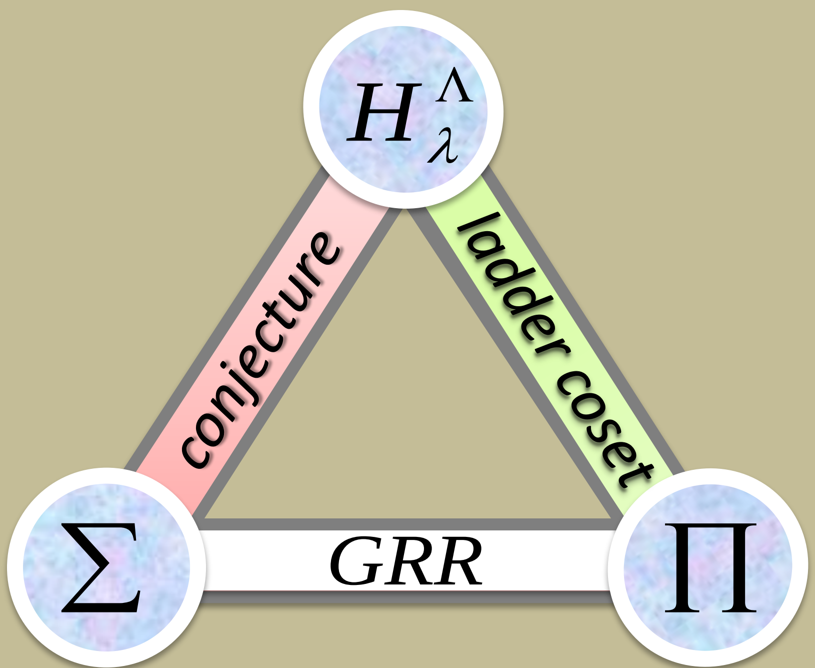

Main Results

- Level two string functions for all simply laced Lie algebras are found. These are the characters of the coset,

$$H_G = G_r / U(1)^r$$

  Where G is any simply laced algebra of rank r at level two.
- An infinite number of conjectured generalized Rogers Ramanujan identities for these characters is described.
- These GRR identities corresponding to A_r and D_r were verified using Mathematica up to $r = 8$.
- For the exceptional algebras E_r all the GRR identities were verified using Mathematica.

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The $H_G = G_r/U(1)^r$ coset.

Characters of the theory H_G, for G_r at any level, are labeled by two weights:

- Highest weight of G_r at level $k = \Lambda$.
- An element of the weight lattice of $G_r = \lambda$.

The weights obey the condition,

$$\Lambda - \lambda = \sum n_i \alpha_i$$

Of particular interest is the parity of n_i ,

$$Q_i \equiv n_i \bmod 2$$

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The general conjecture.

Let Λ_i stand for any fundamental weight with a mark $a_i = 1$.

The characters of the coset H_G are given by,

$$H_\lambda^{\Lambda_i} = q^{\Delta_\lambda^{\Lambda_i}} \sum_{\substack{\vec{b}=0 \\ \vec{b} = \vec{Q} \bmod 2}}^{\infty} \frac{q^{\vec{b} G_r \vec{b}/4 - b_i/2}}{(q)_{\vec{b}}}$$

Where the vector Pochhammer symbol is,

$$(q)_{\vec{b}} = \prod_{j=1}^{r} (q)_{b_j}, \qquad (a)_n = \prod_{l=0}^{n-1} (1 - aq^l)$$

For the A_r and D_r algebras, the case of $\Lambda_i = 0$ was conjectured by A. Kuniba, T. Nakanishi and J. Suzuki(1993).

Ladder coset – quick review

Consider a Lie group with the following property,

$$G_r \supset G_{r-1} \times L_{r-1}, \qquad G_{r-1} \supset G_{r-2} \times L_{r-2} \ldots . G_2 \supset G_1 \times L_1$$

The characters of any of the algebras G_i are given by,

$$G_i{}^{\Lambda_i} = \sum_{\lambda_{i-1}} \sum_{\Lambda_{i-1}} \frac{G_i{}^{\Lambda_i}}{G_{i-1}{}^{\Lambda_{i-1}} L_{i-1}{}^{\lambda_{i-1}}} G_{i-1}{}^{\Lambda_{i-1}} L_{i-1}{}^{\lambda_{i-1}}$$

* Λ_i and λ_i denote DHWs of the G_i and L_i respectively.

* Summation is over DHWs obeying the usual condition.

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Ladder coset – quick review

Specifically, consider the characters of G_r:

$$G_r^{\Lambda_r} = \sum_{\lambda_{r-1}} \sum_{\Lambda_{r-1}} \frac{G_r^{\Lambda_r}}{G_{r-1}^{\Lambda_{r-1}} L_{r-1}^{\lambda_{r-1}}} G_{r-1}^{\Lambda_{r-1}} L_{r-1}^{\lambda_{r-1}}$$

Using the characters decomposition to replace the characters of G_{r-1}, followed by replacing the characters of G_{r-2} ….

$$G_r^{\Lambda_r} = \sum_{\lambda_{r-1}} \cdots \sum_{\lambda_1} \sum_{\Lambda_{r-1}} \cdots \sum_{\Lambda_1} \prod_{i=r-1}^{1} \left(\frac{G_{i+1}^{\Lambda_{i+1}}}{G_i^{\Lambda_i} L_i^{\lambda_i}} \right) L_{r-1}^{\lambda_{r-1}} \ldots L_1^{\lambda_1}$$

$$\Rightarrow \frac{G_r^{\Lambda_r}}{L_{r-1}^{\lambda_{r-1}} \ldots L_1^{\lambda_1} G_1^{\Lambda_1}} = \sum_{\Lambda_{r-1}} \cdots \sum_{\Lambda_2} \prod_{i=r-1}^{1} \left(\frac{G_{i+1}^{\Lambda_{i+1}}}{G_i^{\Lambda_i} L_i^{\lambda_i}} \right)$$

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Outline of proof

- Our objective:
  finding the H_G characters.
- Motivation:

$$c(H_G) = \sum c(L_i)\,,$$

the central charges of the ladder steps theories are smaller. Thus, we are able to identify the ladder steps theories.

ladder construction

$$H_G = \prod L_i$$

↓

identify ladder steps theories L_i

↓

H_λ^Λ = *combinations of known characters*

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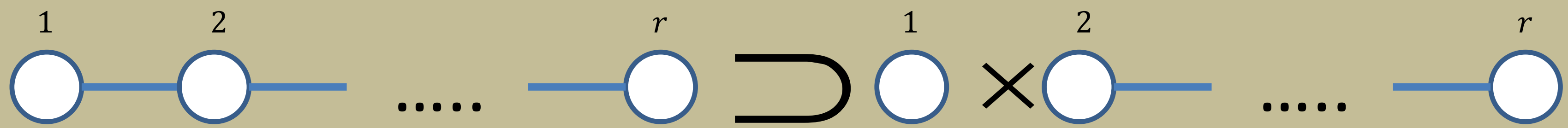
Ladder construction

Consider the classical lie algebras A_r and D_r.

Both have a self contain sub algebra chain:



$$A_r \supset U(1) \times A_{r-1}, \quad A_{r-1} \supset U(1) \times A_{r-2} \quad \ldots \quad A_2 \supset U(1) \times U(1)$$

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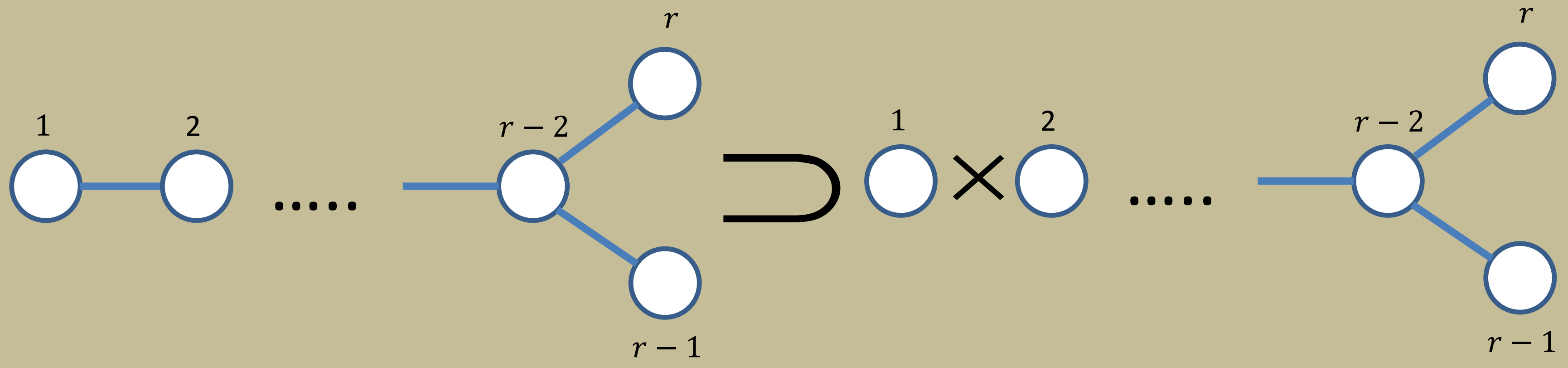
Ladder construction

Consider the classical lie algebras A_r and D_r.

Both have a self contain sub algebra chain:



$$D_r \supset U(1) \times D_{r-1}, \quad D_{r-1} \supset U(1) \times D_{r-2} \quad \dots \quad D_2 \supset U(1) \times U(1)$$

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Ladder construction

Thus, the classical simply laced algebras have a recursive ladder construction:

$$G_r / U(1)^r = \prod_{i=r}^{1} \frac{G_i}{G_{i-1} \times U(1)} \equiv \prod_{i=r}^{1} L(G_i)$$

Where $G_r = A_r, D_r$ and $G_0 = I$.

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Ladder steps theories for A_r

To identify the ladder step theories we invoke level rank duality,

$$\frac{SU(r+1)_2}{SU(r)_2 \times U(1)} \rightarrow \frac{SU(2)_r \times SU(2)_1}{SU(2)_{r+1}} \approx M_{r+2}$$

Which is immediately identified as the minimal model coset representation.

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Characters of $H(A_r)$

Using the ladder coset construction we find

$$SU(r+1)_2/U(1)^r \approx \prod_{i=r}^{1} M_{i+2}$$

Theorem (1) : the $H(A_r)$ coset theory is equivalent to a product theory of r minimal models. Accordingly,

$$H(A_r)_{\lambda}^{\Lambda_r} = \sum_{\Lambda_{r-1}} \dots \sum_{\Lambda_1} \prod_{i=r}^{1} M(i+2)_{\Lambda_{i-1}}^{\Lambda_i}$$

$$\Lambda_i + \Lambda_{i+1} = \lambda_{i+1} \bmod 2$$

- The algebra A_r is discussed in previous work. A. Belavin and D.Gepner GRR identities from AGT correspondence(2013) arXiv:1212.6600 .

Ladder steps theories for D_r

The central charge of the $L(D_r)$ step coset is calculated from coset CFT theory,

$$c\left(D_r / D_{r-1} \times U(1)\right) = 1$$

Due to the classification of $c = 1$ theories we can identify this coset. Calculating the dimensions we find a Z_2 orbifolded free boson,

$$\frac{D_r}{D_{r-1} \times U(1)} \approx B_{orb}\left(R = \sqrt{2r(r-1)}\right)$$

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Characters of $H(D_r)$

Thus, $H(D_r)$ is equivalent to the product theory of r − 1 single boson orbifolds at various radii,

$$SO(2r)_2 / U(1)^r \approx \prod_{i=r-1}^{1} B_{orb}\left(R = \sqrt{2i(i+1)}\right)$$

This implies that the coset theory is equivalent to a theory of orbifolded bosons propagating on some lattice.

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Characters of $H(D_r)$

Using the beta method we identify the lattice as the $SU(r)_2$ lattice.

$$SO(2r)_2 / U(1)^r \approx B_{orb}(SU(r)_2)$$

Theorem (2) : the $H(D_r)$ coset theory is equivalent to a theory of $r-1$ free orbifolded bosons on a $\mathrm{SU(r)}_2$ lattice,

$$H(D_r)^{\Lambda_r}_{\lambda_r} = B_{orb}(SU(r)_2)^{\Lambda_r}_{\lambda_r}$$

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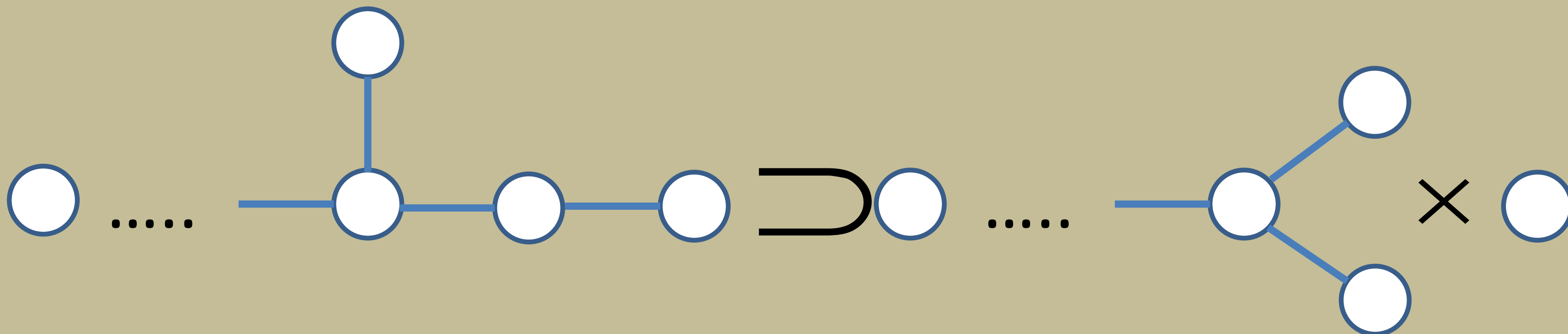
Exceptional algebras ladder construction

All simply laced exceptional algebras emit the subalgebra:



$$E_r \supset D_{r-1} \times U(1)$$

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Exceptional algebras ladder construction

This simplifies the study of H_E,

$$E_r / U(1)^r \approx \frac{E_r}{D_{r-1} \times U(1)} \times \frac{D_{r-1}}{U(1)^{r-1}}$$

Using **theorem(2)** we find the H_E coset theory is given by the product theory of $r-2$ free orbifolded bosons on a $\mathrm{SU}(\mathrm{r}-1)_2$ lattice and L_{E_r}.

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The H_{E_6} theory

The L_{E_6} coset theory is identified by its central charge,

$$c\left(E_6/D_5 \times U(1)\right) = 8/7$$

This is the central charge of a level five parafermion.

Theorem (3) : the H_{E_6} theory is given by a product of 4 orbifolded bosons on a $SU(5)_2$ lattice and a level five parafermion oribfolded by the symmetry $A^\dagger \to A$

$$H(E_6)^{\Lambda_6}_{\lambda_6} = \sum_{\mu_5} \chi_5{}^{\Lambda_6}_{\lambda_1,\mu_5} \times B_{orb}(SU(5)_2)^{\mu_5}_{\lambda_5}$$

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The H_{E_7} theory

The central charge of the L_{E_7} coset theory,

$$c\left(E_7/D_6\times U(1)\right)=13/10=1/2+4/5$$

Which is the central charge of the product of minimal models $M_3\times M_5$.

Theorem (4) : the H_{E_7} theory is given by the product theory of 5 orbifolded bosons on a $SU(6)_2$ lattice and the $M_3\times M_5$ theory .

$$H(E_7)_{\lambda_7}^{\Lambda_7}=\sum_{\mu_6}(M_3\times M_5)_{\lambda_1}^{\Lambda_7}\times B_{orb}(SU(6)_2)_{\lambda_6}^{\mu_6}$$

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The H_{E_8} theory

The E_8 ladder construction is simplified by use of the coset,

$$E_8/D_8 \approx M_3 \quad \Rightarrow \quad E_8/U(1)^8 \approx M_3 \times \frac{D_8}{U(1)^8}$$

Theorem (5) : the H_{E_8} theory is given by the product theory of 7 orbifolded bosons on a $SU(8)_2$ lattice and the Ising model,

$$H(E_8)^{\Lambda_8}_{\lambda_8} = \sum_{\mu_8} M_3 {}^{\Lambda_8}_{\mu_8} \times B_{orb}(SU(8)_2)^{\mu_8}_{\lambda_8}$$

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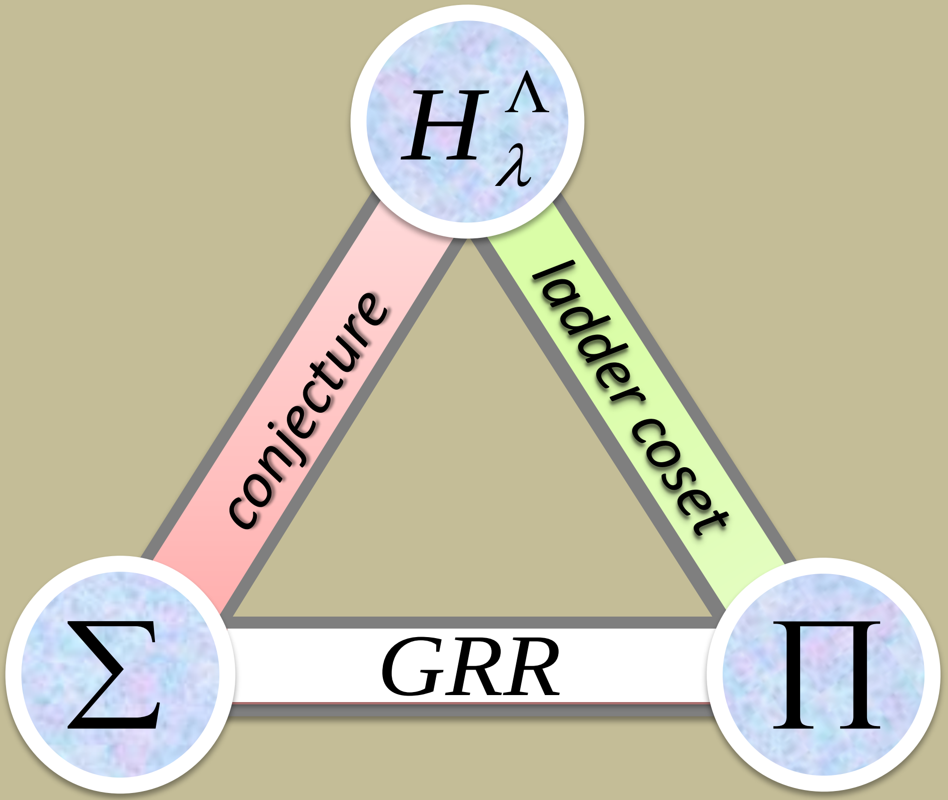
Generalized Roger Ramanujan identities

- GRR identities are generally of the $\Sigma = \Pi$ form.
- The characters found by the ladder coset are of the Π type.
- Our conjecture is of the GRR Σ type.

New GRR Identities



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Generalized Roger Ramanujan identities

- Using the H_G characters we verified our conjectured GRR identities.
- For example, we conjecture that the singlet character of H_G at level two is:

$$H^0_0 = q^{-(r-1)/24} \sum_{\substack{\vec{b}=0 \\ \vec{b} = 0 \bmod 2}}^{\infty} \frac{q^{\vec{b} G_r \vec{b}/4}}{(q)_{\vec{b}}}$$

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New GRR from D_r algebra

- On the other hand, the singlet character is given by the various theorems.
- For the D_r algebra, the singlet character leads to the GRR identity:

$$\frac{1}{(q)_\infty^{\ r-1}} \sum_{\vec{n}=-\infty}^{\infty} q^{\vec{n} A_{r-1} \vec{n}} + \frac{1}{(-q)_\infty^{\ r-1}} = 2 \sum_{\substack{\vec{b}=0 \\ \vec{b}\ =\ 0 \bmod 2}}^{\infty} \frac{q^{\vec{b} D_r \vec{b}/4}}{(q)_{\vec{b}}}$$

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New GRR from D_r algebra

- Other particularly beautiful identities arise from the twisted sector,

$$(q^{1/2})_{\infty}^{1-r} \pm (-q^{1/2})_{\infty}^{1-r} = 2q^{\Delta_{\pm}} \sum_{\substack{\vec{b}=0 \\ b_i = 0 \bmod 2 \\ b_1 = Q_{\pm} \bmod 2}}^{\infty} \frac{q^{\vec{b} D_r \vec{b}/4 - b_{r/2}}}{(q)_{\vec{b}}}$$

- Where $Q_{+} = 0$ and $Q_{-} = 1$.

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New GRR from E_8 algebra

- Consider the field ${H_\lambda}^0$ of the E_8 coset theory where $\lambda = s, v + \bar{s}$ of D_8.
- Our conjecture for the character ${H_\lambda}^0$ is,

$$H_\lambda^0 = q^{\Delta_\pm} \sum_{\substack{\vec{b}=0 \\ \vec{b} = \vec{Q}_\pm \bmod 2}}^{\infty} \frac{q^{\vec{b} E_8 \vec{b}/4}}{(q)_{\vec{b}}}$$

- By theorem(5) the character is given by,

$$H_\lambda^0 = M_{3s}^{\;0} \times B_\lambda^s$$

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New GRR from E_8 algebra

- The Ising model character is identified from its dimension, $\Delta({E_8}^0/{D_8}^s) = 1/16$

$$M_{3\ s}^{\ \ 0} = q^{1/16}(-q)_\infty$$

- While ${B^s}_\lambda$ are the twisted sector characters of the bosonic $SU(8)_2$,

$$B_\lambda^s = \frac{1}{2}(q^{1/2})_\infty^{-7} \pm \frac{1}{2}(-q^{1/2})_\infty^{-7}$$

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New GRR from E_8 algebra

- The conjectured GRR identities, $H_\lambda^0 = M_{3_s}^{\ 0} \times B_\lambda^s$:

$$q^{\Delta_\pm} \sum_{\vec{b}=0}^{\infty} \frac{q^{\vec{b} E_8 \vec{b}/4}}{(q)_{\vec{b}}} = \frac{1}{2}(-q)_\infty \left((q^{1/2})_\infty^{-7} \pm (-q^{1/2})_\infty^{-7} \right)$$

- Where the sum is restricted by,

$$\vec{b} = \vec{Q}_\pm \mod 2 \qquad \begin{aligned} Q_+ &= (1,0,0,0,0,0,0,0) \\ Q_- &= (1,0,0,0,0,1,0,0) \end{aligned}$$

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Discussion

- String functions were found for all simply laced algebras of level 2 and rank r.

- These are the H_G characters (of all DHWs) where G is any simply laced algebra of level 2 and rank r.

- New general theta function identities are found through symmetries of the ladder construction.

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Discussion

- A conjectured GRR sum type expression was given for characters of fundamental weights with mark one.
- Combined with the exact characters a conjecture for an infinite number of new GRRs was found,

$$H_{\lambda}^{\Lambda_i} = q^{\Delta_{\lambda}^{\Lambda_i}} \sum_{\substack{\vec{b}=0 \\ \vec{b} = \vec{Q} \bmod 2}}^{\infty} \frac{q^{\vec{b} G_r \vec{b}/4 - b_i/2}}{(q)_{\vec{b}}}$$

- These GRR identities corresponding to A_r and D_r were verified using Mathematica up to $r = 8$.
- For the exceptional algebras all the GRR identities were verified using Mathematica.

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Current research

- We have proven the conjecture for $SO(6)_2$ and $SO(8)_2$:
  - New GRR identities and partition theories.
  - Generalized conjecture for all DHWs.
  - New solvable IRF models?
- General proof through IRF models one point functions is an intriguing possibility which we are currently investigating.

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