Scaling limit of the staggered six-vertex model

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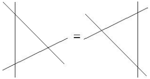
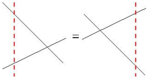
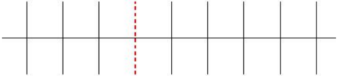
work with Alexander Seel, U Siegen

Inhmogeneities & staggering





class of Lax-operators with identical R-matrix



$\succ$ commuting transfer matrices with inhomogeneities

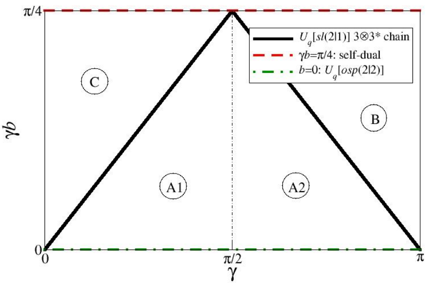
Examples / Applications:

- quantum impurities:
  Kondo/Anderson impurity problems [Andrei, Furuya, Lowenstein (1983); Tsvelik, Wiegmann (1983)]
  dynamical (boundary) impurities in lattice models
  [Andrei, Johannesson (1984); Bedürftig, Essler, HF (1997); HF, Zvyagin (1997); Zhou *et al.* (1999); HF, Slavnov (1999)]
- staggered six-vertex model:
  equivalence of square lattice Potts model to staggered six-vertex model
  [Temperley, Lieb (1971); Baxter, Kelland, Wu (1976)]
  spin chains with n-neighbour interactions [Popkov, Zvyagin (1993); HF, Rödenbeck (1996)]
  thermodynamics via quantum transfer matrix [Klümper (1993)]
- staggered superspin chains:
  integrable 'network' models for disorder problems (e.g. QHE) as vertex models with alternating representations of superalgebra
  [Gade (1999); Gruzberg *et al.* (1999); Essler, HF, Saleur (2005); HF, Martins (2011,2012)]

Critical properties of alternating superspin chains

$U_q[sl(2|1)]$ **vertex model:**
alternating quark-antiquark ($3 \otimes \bar{3}$) / 4d-typical rep ($[b, \frac{1}{2}] \otimes [-b, \frac{1}{2}]$)

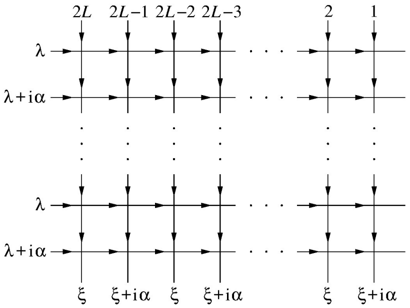
- contains staggered 6v model in zero charge sector
- $\gamma b = \pi/4$: $\mathbb{Z}_2$-symmetry as consequence of self-duality under $\gamma b \leftrightarrow \frac{\pi}{2} - \gamma b$: twisted $U_q[D_2^{(2)}]$ quantum group symmetry, $q = e^{i\gamma}$ (Potts model in zero charge sector)



- phases A1, A2: two (compact) bosonic degrees of freedom (spin and charge)
- phases B ($b > \frac{1}{2}$) and C ($b \geq \frac{1}{2}$): in addition one compact degree of freedom (charge resp. spin) plus a **continuum of critical exponents** [Essler, Hf, Saleur (2005); HF, Martins (2012)]

The $\mathbb{Z}_2$ staggered six-vertex model

We consider the spin sector of the $U_q[sl(2|1)]$ 4-state model in phase B (i.e. $\gamma < \alpha = 2\gamma b < \pi - \gamma$ in $\gamma \in (0, \pi/2)$):



local Boltzmann weights given by R-matrix of the six-vertex model

- self-dual for $\alpha = \pi/2$: Potts model
- double-row transfer matrix $t^{(2)}(\lambda) = t(\lambda)\, t(\lambda + i\alpha)$: two-site shift at $\lambda = \xi$
- local Hamiltonian $H = -i\partial_\lambda \log(t^{(2)}(\lambda))|_{\lambda=\xi}$.

Bethe ansatz solution

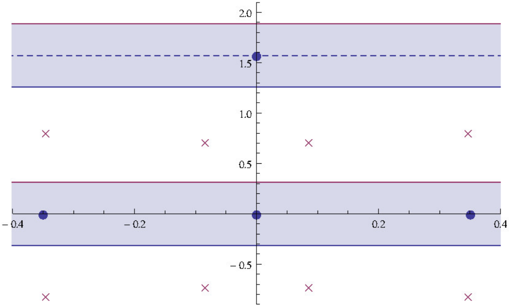
Spectrum of $t^{(2)}$ is parameterized by parameters $\{\lambda_j\}_{j=1}^M$:

$$\left(\frac{\mathrm{sh}(\lambda_j+\frac{i\alpha}{2}+\frac{i\gamma}{2})}{\mathrm{sh}(\lambda_j+\frac{i\alpha}{2}-\frac{i\gamma}{2})}\frac{\mathrm{sh}(\lambda_j-\frac{i\alpha}{2}+\frac{i\gamma}{2})}{\mathrm{sh}(\lambda_j-\frac{i\alpha}{2}-\frac{i\gamma}{2})}\right)^L=\prod_{\substack{\ell=1\\ \ell\neq j}}^{M}\frac{\mathrm{sh}(\lambda_j-\lambda_\ell+i\gamma)}{\mathrm{sh}(\lambda_j-\lambda_\ell-i\gamma)}\,.$$

Low energy root configurations:

- n_1 real roots
- n_2 roots with $Im(\lambda_j)=\frac{\pi}{2}$
- $n_1+n_2\approx L$

e.g. $L=4,\ n_1=3,\ n_2=1\rightarrow$



Finite size spectrum

thermodynamic limit: ground state has L roots ($S^z = 0$), densities:

$$\frac{n_1^{(0)}}{L} = \frac{\pi - \alpha - \gamma}{\pi - 2\gamma} = 1 - \frac{n_2^{(0)}}{L}\,.$$

Root density approach for states with $n_{1,2} = n_{1,2}^{(0)} + \frac{1}{2}\left(m \pm \widetilde{m}\right)$ and vorticity w gives [Jacobsen, Saleur (2006); Ikhlef, Jacobsen, Saleur (2008); HF, Martins (2012)]

$$E(L) - L\varepsilon_\infty = \frac{2\pi\, v_F}{L}\left(-\frac{1}{6} + \frac{\gamma}{2\pi}\, m^2 + \frac{\pi}{2\gamma}\, w^2 + \kappa(L)\,\widetilde{m}^2 + \ldots\right)$$

with $\kappa(L) \propto (\log L)^{-2}$ for $\gamma < \frac{\pi}{2}$ based on numerical evidence. Root density approach plus Wiener-Hopf methods for self-dual case gives [Ikhlev, Jacobson, Saleur (2012)]

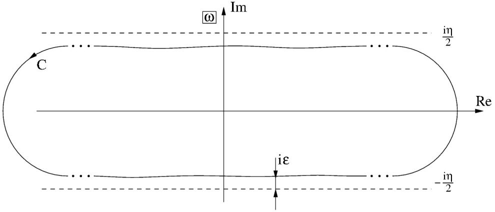
$$\kappa(L) = \frac{\pi^2\gamma}{8(\pi - 2\gamma)}\,\frac{1}{\log L^2}$$

Nonlinear integral equations

Analysis of logarithmic finite size corrections requires *very* large lattice sizes: encode position of Bethe roots in solutions to NLIEs ($n_1 + n_2 = L$, $\gamma' = \frac{\pi}{2} - \gamma$)

$$\log \mathfrak{a}_1(\lambda) = 2i\gamma L + L \log\left(\frac{\mathrm{sh}(\lambda + \frac{i\alpha}{2} - \frac{i\gamma}{2})}{\mathrm{sh}(\lambda + \frac{i\alpha}{2} + \frac{i\gamma}{2})} \frac{\mathrm{sh}(\lambda - \frac{i\alpha}{2} - \frac{i\gamma}{2})}{\mathrm{sh}(\lambda - \frac{i\alpha}{2} + \frac{i\gamma}{2})} \right)$$
$$- \int_{\mathcal{C}} \frac{\mathrm{d}\omega}{2\pi} K_{i\gamma}(\lambda - \omega) \log(1 + \mathfrak{a}_1)(\omega) + \int_{\mathcal{C}} \frac{\mathrm{d}\omega}{2\pi} K_{i\gamma'}(\lambda - \omega) \log(1 + \mathfrak{a}_2)(\omega)$$

and similar for $\mathfrak{a}_2(\lambda) = \mathfrak{a}_1(\lambda + i\pi/2)$ plus energy etc. in terms of $\mathfrak{a}_k$.
Integration is performed along contour $\mathcal{C}$



$\eta = \min\{\gamma, \alpha - \gamma\}$

→ Poster by Alexander Seel.

Fine structure of the finite size spectrum

Low energy states are characterized by conserved quantum numbers

- $m \succ$ magnetization S^z_{tot}
- $w \succ$ vorticity / total momentum
- $\widetilde{m}$: **???**

$\widetilde{m} \neq 0$ reflects asymmetry in no. of roots on $Im(\lambda_j) = 0, \pi/2$: can be measured by *quasi-shift operator* $\widetilde{\tau} \equiv t(\xi)\,[t(\xi + i\alpha)]^{-1}$ and corresponding *quasi-momentum*
[Ikhlef, Jacobsen, Saleur (2012); Candu, Ikhlef (2013)]

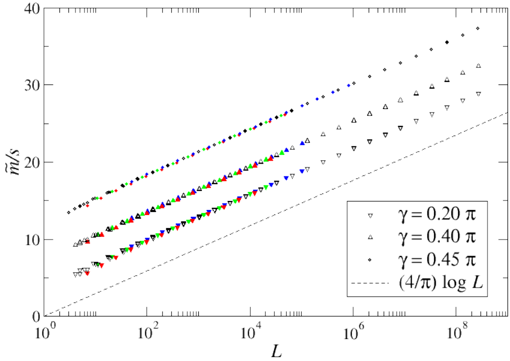
$$\widetilde{P} \equiv \log\left(t(\xi)\,[t(\xi + i\alpha)]^{-1}\right)$$

with eigenvalues K. By construction $\widetilde{P}$ commutes with the Hamiltonian.
Continuous quantum number for continuous part of spectrum
[Ikhlef, Jacobsen, Saleur (2012); HF, Seel (2014)]

$$s \equiv \frac{\pi - 2\gamma}{4\pi\gamma}\left(K - Lk_\infty\right), \qquad \widetilde{m} \simeq \frac{4s}{\pi}\left(\log\frac{L}{L_0} + B(s)\right).$$

k_∞ is quasi-momentum density in thermodynamic limit (= 0 for self-dual model)

Fine structure of the finite size spectrum



- independent of staggering α (different colors)
- γ-dependent offset $\succ L_0(\gamma)$
- density of states in continuum ($\Delta \widetilde{m} = 2$):

$$\rho(s) = \frac{1}{2}\partial_s \widetilde{m} = \frac{2}{\pi}\left(\log\frac{L}{L_0} + \partial_s(sB(s))\right)$$

Conformal field theory

$$\Delta E = \frac{2\pi v_F}{L}\left(-\frac{c}{12} + h + \bar{h}\right), \quad P = \frac{2\pi}{L}\left(h - \bar{h}\right).$$

- ground state of the lattice model ($(m, w) = (0, 0)$ and $\widetilde{m} = 0$):

$$c_{\text{eff}} = 2$$

- continuous quantum no. $s \succ$ indication for non-rational CFT with non-normalizable Virasoro vacuum: state $h = \bar{h} = 0$ not in spectrum of lattice model!
- proposal for self-dual case [Ikhlef, Jacobsen, Saleur (2008)]:

  $SL(2,\mathbb{R})/U(1)$ σ-model at level $k = \frac{\pi}{\gamma} \in (2, \infty) \succ$ Euclidean black hole:

$$c = 2\frac{k+1}{k-2} = 2 + \frac{6}{k-2},$$

$$h = \frac{(m-kw)^2}{4k} + \frac{s^2+1/4}{k-2}, \quad \bar{h} = \frac{(m+kw)^2}{4k} + \frac{s^2+1/4}{k-2}$$

Conformal field theory

- vacuum not normalizable: ground state of lattice model $\equiv$ lowest conformal weight:

$$h_0 = \bar{h}_0 = \frac{1}{4(k-2)} \quad \succ \quad c_{\text{eff}} = 2$$

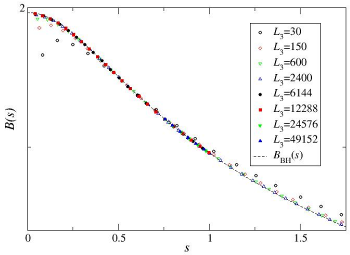
- density of states, computed from finite part $B(s)$ in continuum agrees with that of the black hole σ-model

[Maldacena, Ooguri, Son (2001); Ikhlef, Jacobsen, Saleur (2012); Candu, Ikhlef (2013); HF, Seel (2014)]

$$\rho_{BH}(s) = \frac{2}{\pi}\Big(\log\epsilon + \partial_s(s\, B_{BH}(s))\Big)$$

$$B_{BH}(s) = \frac{1}{s}\,\text{Im}\log\Gamma(\frac{1}{2} - is)$$

$$\gamma = \frac{\pi}{5},\, \alpha = \frac{2\pi}{5} \;\rightarrow$$



Corrections to scaling

Additional information on the model can be obtained from *subleading corrections* to the finite size spectrum:

$$E_a(L) - L\varepsilon_\infty = \frac{2\pi v_F}{L}\left(-\frac{c}{12} + (h_a + \bar{h}_a) + R_a(L)\right)$$

due to deviations of the lattice model from the conformally invariant fixed point

$$H_{\text{lattice}} = H^* + \sum_b g_b \int \mathrm{d}x\, \Phi_b(x)$$

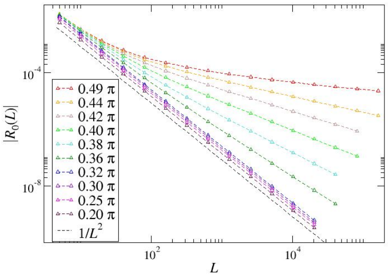
2^{nd} order perturbation theory gives [Cardy 1986; Alcaraz,Barber, Batchelor (1988)]

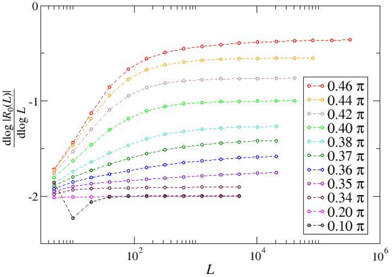
$$R_a(L) \simeq 2\pi \sum_b g_b\, C_{a,a,b} \left(\frac{2\pi}{L}\right)^{X_b-2} + 4\pi^2 \sum_{\substack{b,b',a'\\ a'\neq a}} g_b g_{b'} \frac{C_{a,a',b}\, C_{a',a,b'}}{X_a - X_{a'}} \left(\frac{2\pi}{L}\right)^{X_b+X_b'-4} + \ldots$$

with scaling dimensions $X_b = h_b + \bar{h}_b$ and OPE coefficients $C_{a,b,c}$.

Corrections to scaling

Numerical data for L up to 10^6:





Asymptotically, scaling corrections in the ground state vanish as power of system size $R_0(L) \sim L^{-\delta}$:

$$\delta = \begin{cases} 2 & \text{for} \quad 0 < \gamma < \frac{\pi}{3} \\ \frac{2\pi}{\gamma} - 4 & \text{for} \quad \frac{\pi}{3} \leq \gamma < \frac{\pi}{2} \end{cases}$$

Corrections to scaling

- 'analytic' corrections ($\delta = 2$) due to perturbation by descendent of identity operator with $X_{\mathbb{I}} = 4$
- slow crossover from $\delta = 2$ to $\frac{2\pi}{\gamma} - 4$ with L: indication of continuous spectrum of perturbing operators with dimension starting at

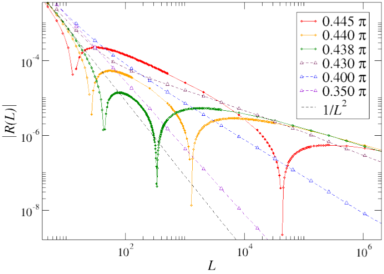
$$X_k = \delta + 2 = \frac{2\pi}{\gamma} - 2 = 2(k-1)$$

- $\delta \to 0$, $X_k \to 2$ as $\gamma \to \frac{\pi}{2}$:
  perturbation drives system to different low-E theory (as expected from $OSP(2|2)$ symmetry in the vertex model): "marginal"

This is independent of the staggering α. Similar scaling corrections are observed for excited states, e.g.

$$(m, w) = (0, 0)\,, \quad \widetilde{m} = 1$$

same asymptotic scaling behaviour but oscillations (and sign changes) of $R(L)$ in the crossover $\delta = 2$ to $X_k - 2$.



Corrections to scaling

Operator content of $SL(2,\mathbb{R})/U(1)$ coset model:

$$X_{(m,w)}(s) = \frac{m^2}{2k} + \frac{kw^2}{2} + \frac{2s^2 + \frac{1}{2}}{k-2}$$

$\succ$ **no field** with scaling dimension $X_k = 2(k-1)$:
contribution $2k$ from fields with vorticity $w = \pm 2$, but non-compact degrees of freedom should give divergence as $k \to 2$ ($\gamma \to \pi/2$).

Possible sources of additional L-dependence:
Assume that the least irrelevant perturbing operators are

- descendent of identity
- operators from the continuum of fields with $(m,w) = (0,2)$ and $SL(2,\mathbb{R})$-spin $j = -\frac{1}{2} + is$, quantized on the lattice by $\Delta s \simeq \pi/(2\log L)$.

Therefore the OPE coefficients in the first order expressions for $R_0(L)$ are

$$C_{(0,0),(0,0),(0,2)}\left(s_1, s_2 = 0, s_3 = \frac{\pi n}{2\log L}\right), \quad n = 0,1,2,\ldots$$

C's are double Gamma functions depending on s_a and k ...

Summary & Outlook

- finite size analysis of the staggered six-vertex model for parameters $0 \leq \gamma < \alpha < \pi - \gamma$ using Bethe ansatz and NLIE for $L \gtrsim 10^6$.
- continuous quantum number for non-compact degree of freedom can be related to *quasi momentum* of lattice model.
- finite size spectrum and density of states: critical theory has been identified as $SL(2,\mathbb{R})/U(1)$ sigma model at level $k = \frac{\pi}{\gamma}$ — independent of α.
- subleading corrections to scaling are power laws, depending on anisotropy, as consequence of perturbation by continuum of operators.
- transition to different theory as $\gamma \to \frac{\pi}{2}$
  — but not due to presence of marginal ($X = 2$) operator in the CFT!
  needs to be taken into account in finite size analysis

To do:

- quantitative analysis of scaling corrections, in particular modified L-dependence requires better understanding of OPE/lattice regularization for theories with non-compact target space.
- extension of NLIE to states with non-zero magnetization, i.e. $L \neq n1 + n2$.
- extension to susy staggered vertex models

H. Frahm and A. Seel, *Nucl. Phys. B* **879** (2014) 382-406, arXiv:1311.6911