Bilinear equations on Painlevé τ-functions from CFT

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Painleve equations

- Painlevé VI is the most general equation of type $q'' = F(t, q, q')$ without movable critical points except poles.

$$\frac{d^2q}{dt^2} = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\right)\frac{dq}{dt} +$$
$$+ \frac{2q(q-1)(q-t)}{t^2(t-1)^2}\left(\left(\theta_\infty - \frac{1}{2}\right)^2 - \frac{\theta_0^2 t}{q^2} + \frac{\theta_1^2(t-1)}{(q-1)^2} - \frac{\left(\theta_t^2 - \frac{1}{4}\right)t(t-1)}{(q-t)^2}\right).$$

2nd order, 4 parameters $\vec{\theta} = (\theta_0, \theta_t, \theta_1, \theta_\infty)$

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- Painlevé III_3'

$$\frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{2q^2}{t^2} + \frac{2}{t},$$

Isomonodromic deformations

- Linear system of rank N with n regular singularities $a = \{a_1, \ldots, a_n\}$ on $\mathbb{CP}^1$:

$$\partial_z \Phi = \mathcal{A}(z)\Phi, \qquad \mathcal{A}(z) = \sum_{\nu=1}^{n} \frac{\mathcal{A}_\nu}{z - a_\nu},$$

$\mathcal{A}_\nu \in \mathfrak{sl}_N$, $\quad \sum_{\nu=1}^{n} \mathcal{A}_\nu = 0$, $\quad \mathcal{A}_\nu = \mathcal{G}_\nu \mathcal{T}_\nu \mathcal{G}_\nu^{-1}$, where
$\mathcal{T}_\nu = \mathrm{diag}\left\{\lambda_{\nu,1}, \ldots, \lambda_{\nu,N}\right\},$ $\quad$ Non-resonance condition: $\lambda_{\nu,j} - \lambda_{\nu,k} \notin \mathbb{Z}$.

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- Let us now vary the positions of singularities a_ν simultaneously evolving $\mathcal{A}_\nu$'s in such way that the monodromy is preserved.
  Schlesinger deformation equations $\Rightarrow d\left(\sum_{\mu<\nu}\mathrm{Tr}\,\mathcal{A}_\mu\mathcal{A}_\nu\, d\ln\left(a_\mu-a_\nu\right)\right)=0$.

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Schlesinger deformation equations $\Rightarrow d\left(\sum_{\mu<\nu} \mathrm{Tr}\, \mathcal{A}_\mu \mathcal{A}_\nu \; d \ln (a_\mu - a_\nu)\right) = 0$.
- Isomonodromic τ is defined on universal covering of $\mathbb{C}^n \setminus \bigcup_{i \neq j} \{a_i - a_j\}$ by

$$d \ln \tau(a_1, \dots, a_n) = \sum_{\mu<\nu} \mathrm{Tr}\, \mathcal{A}_\mu \mathcal{A}_\nu \; d \ln (a_\mu - a_\nu).$$

τ function depend on parameters — monodromy date.

$\mathcal{M}_{0,4,\theta}$

- Painlevé VI correspond to the rank 2 linear systems on $\mathbb{CP}^1$ with 4 regular singular points $0, t, 1, \infty$. Eigenvalues of A_ν are $(\theta_\nu, -\theta_\nu)$, $\nu = 0, 1, t, \infty$

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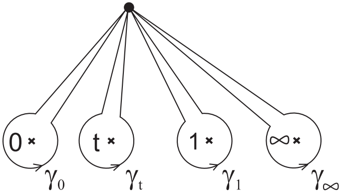
- Monodromy matrices $M_0, M_t, M_1, M_\infty \in SL_2(\mathbb{C})$.



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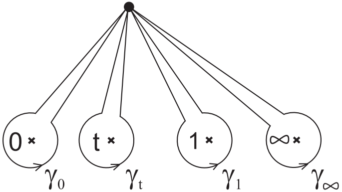

- The module space of flat rank 2 connections:

$$\mathcal{M}_{0,4,\theta} = \left\{ M_\nu \middle| \begin{array}{l} \mathrm{tr} M_\nu = 2\cos(2\pi\theta_\nu), \\ M_\infty M_1 M_t M_0 = 1 \end{array} \right\} \Big/ SL_2$$

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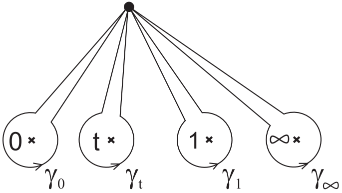

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- As coordinates on $\mathcal{M}_{0,4,\theta}$ one can use $\sigma_{\mu\nu}$ defined by $\mathrm{tr} M_\mu M_\nu = 2\cos(2\pi\sigma_{\mu\nu})$, $\mu, \nu = 0, t, 1$.

Coordinates on $\mathcal{M}_{0,4,\theta}$

- The manifold $\mathcal{M}_{0,4,\theta}$ defied by Fricke-Jimbo relation $W(p_{0t}, p_{1t}, p_{01}) = 0$:

$$W(p_{0t}, p_{1t}, p_{01}) = p_{0t}p_{1t}p_{01} + p_{0t}^2 + p_{1t}^2 + p_{01}^2 - p_{0t}(m_0 m_t + m_1 m_\infty) -$$
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$$s_{0t}^{\pm 1} = \frac{\frac{\partial W}{\partial p_{1t}} e^{\pm 2\pi i \sigma_{0t}} + \frac{\partial W}{\partial p_{1t}}}{16 \prod_{\epsilon=\pm} \sin\pi(\theta_t \mp \sigma_{0t} + \epsilon\theta_0) \sin\pi(\theta_0 \mp \sigma_{0t} + \epsilon\theta_\infty)}.$$

This coordinates $\sigma, \log s$ are Darboux coordinates. Them are closely related to the Nekrasov, Rosly Shatashvili coordinates α, β.

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- For the Painlevé VI equations σ, s are the integration constants.

$t = 0$ expansion

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- Gamayun, Iorgov, Lisovyy (2012) conjecture

Theorem

The expansion Painlevé VI τ function near $t = 0$ can be written as

$$\tau(\overrightarrow{\theta}, s, \sigma; t) = \sum_{n \in \mathbb{Z}} C(\sigma + n, \overrightarrow{\theta}) \cdot s^n \mathcal{F}(\overrightarrow{\Delta}, (\sigma + n)^2 | t)$$

$\mathcal{F}(\overrightarrow{\Delta}, (\sigma + n)^2, |t)$ — 4-point $\mathbb{CP}^1$ conformal block for central charge $c = 1$.

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- Generalize Jimbo assymtotic formuala
  Applications: expansions and connection.
  Another proof: by Iorgov, Lisovyy, Teschner 2014.

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The expansion Painlevé III_3' τ function near $t = 0$ can be written as

$$\tau(\sigma, s; t) = \sum_{n\in\mathbb{Z}} C(\sigma+n)s^n\mathcal{F}((\sigma+n)^2|t),$$

$\mathcal{F}(\sigma^2|t)$ —irregular (Whittaker) limit of Virasoro (Vir) conformal block for central charge $c = 1$. The coefficients $C(\sigma) = 1/\Big(\mathsf{G}(1-2\sigma)\mathsf{G}(1+2\sigma)\Big)$.

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- Our proof is based on the bilinear equation on τ functions :

$$D^{III}(\tau(t),\tau(t))=0,\quad \text{where}\quad D^{III}=\frac{1}{2}D^4_{[\log t]}-t\frac{d}{dt}D^2_{[\log t]}+\frac{1}{2}D^2_{[\log t]}+2tD^0_{[\log t]},$$

Here $D^k_{[\log t]}$ is a Hirota differential operator defined by:

$$f(e^\alpha t)g(e^{-\alpha}t)=\sum_{k=0}^{+\infty}D^k_{[\log t]}(f(t),g(t))\frac{\alpha^k}{k!}$$

Bilinear conformal blocks relations

- We substitute $\tau(t)$ to the equation and collect terms with the same powers of s. The vanishing condition of s^m coefficient have the form:

$$\sum_{n\in\mathbb{Z}} C(\sigma+n+m)C(\sigma-n) \times D^{III}\left(\mathcal{F}((\sigma+n+m)^2|t), \mathcal{F}((\sigma-n)^2|t\right) = 0,$$

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- Each summand — $\mathrm{Vir} \oplus \mathrm{Vir}$.
  The whole sum — $\mathrm{Vir} \oplus \mathrm{Vir} \subset \mathrm{F} \oplus \mathrm{NSR}$,
  where F is a Majorana fermion algebra and NSR is Neveu–Schwarz–Ramond algebra, $\mathcal{N}=1$ superanalogue of the Virasoro algebra.

Instanton counting

- Let $M(\mathbb{C}^2; r, N)$ denotes the moduli space of instantons on $\mathbb{C}^2$ (of rank r, $c_2 = N$). By $Z(\epsilon_1, \epsilon_2, a; q)$ we denote Nekrasov instanton partition function for the pure $U(2)$ gauge theory on $\mathbb{C}^2$: $Z(\epsilon_1, \epsilon_2, a; q) = \sum_{N=0}^{\infty} q^N \int\limits_{M(\mathbb{C}^2; r, N)} 1.$

  Due to AGT relation $Z(\epsilon_1, \epsilon_2, a; q)$ correspond to irregular confromal block with $c = 1 + 6\frac{(\epsilon_1+\epsilon_2)^2}{\epsilon_1\epsilon_2}$.

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- Nakajima-Yoshioka blow-up equations (2003):

$$Z(\epsilon_1, \epsilon_2, a; q) = \sum_{n\in\mathbb{Z}} \frac{q^{k^2}}{\boldsymbol{l}_n(a, \epsilon_1, \epsilon_2)} Z(\epsilon_1, \epsilon_2-\epsilon_1, a+n\epsilon_1; q)\cdot Z(\epsilon_1-\epsilon_2, \epsilon_2, a+n\epsilon_2; q),$$

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  Impossible to make $c^{(1)} = c^{(1)} = 1$.
- X_2— minimal resolution of $\mathbb{C}^2/\mathbb{Z}_2$. Bonelli, Maruyoshi, Tanzini (2012):

$$Z^{X_2}(\epsilon_1, \epsilon_2, a; q) = \sum_{2n\in\mathbb{Z}} \frac{q^{2n^2}}{\boldsymbol{l}_n(a, \epsilon_1, \epsilon_2)} \cdot Z(2\epsilon_1, \epsilon_2 - \epsilon_2, a + 2n\epsilon_1; q) Z(\epsilon_1 - \epsilon_2, 2\epsilon_2, a + 2n\epsilon_2; q)$$

  If $\epsilon_1 + \epsilon_2 = 0$ then $c^{(1)} = c^{(2)} = 1$.

  The coefficients $1/\boldsymbol{l}_n(a, \epsilon_1, \epsilon_2)$ are proportional to $\dfrac{C(\sigma + n)C(\sigma - n)}{C(\sigma)^2}$.

The F ⊕ NSR algebra

- The F ⊕ NSR algebra is a direct sum of the free-fermion algebra F with generators f_r ($r \in \mathbb{Z} + \frac{1}{2}$) and NSR (Neveu-Schwarz-Ramond or Super Virasoro) algebra with generators L_n, G_r ($n \in \mathbb{Z}, r \in \mathbb{Z} + \frac{1}{2}$).

$$\{f_r, f_s\} = \delta_{r+s,0}, \qquad [L_n, L_m] = (n-m)L_{n+m} + \frac{c_{\text{NSR}}}{8}(n^3 - n)\delta_{n+m},$$
$$[L_n, G_r] = \left(\frac{1}{2}n - r\right) G_{n+r}, \qquad \{G_r, G_s\} = 2L_{r+s} + \frac{c_{\text{NSR}}}{2}(r^2 - \frac{1}{4})\delta_{r+s,0}.$$

The central charge c_{NSR} is parameterized as: $c_{\text{NSR}} = 1 + 2(b + b^{-1})^2$.

The F $\oplus$ NSR algebra

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- Two commuting Virasoro (Crnkovic et al 1989; Lashkevich 1993)

$$L_n^{(1)} = \frac{1}{1-b^2}L_n - \frac{1+2b^2}{2(1-b^2)} \sum_{r=-\infty}^{\infty} r : f_{n-r}f_r : + \frac{b}{1-b^2} \sum_{r=-\infty}^{\infty} f_{n-r}G_r,$$

$$L_n^{(2)} = \frac{1}{1-b^{-2}}L_n - \frac{1+2b^{-2}}{2(1-b^{-2})} \sum_{r=-\infty}^{\infty} r : f_{n-r}f_r : + \frac{b^{-1}}{1-b^{-2}} \sum_{r=-\infty}^{\infty} f_{n-r}G_r,$$

The central charges: $c^{(1)} = 1 + \frac{3(b+b^{-1})^2}{(b^2-1)}$, $c^{(2)} = 1 + \frac{3(b+b^{-1})^2}{(b^{-2}-1)}$.

Verma modules

- Denote Verma module of Vir as $\pi_{\mathrm{Vir}}^{\Delta}$. This module has highest weight vector $|\Delta\rangle$, defined by properties:

$$L_0|\Delta\rangle = \Delta|\Delta\rangle, \quad L_n|\Delta\rangle = 0, \quad n > 0,$$

We denote $\pi_{\mathrm{F}\oplus\mathrm{NSR}}^{\Delta^{\mathrm{NS}}}$ — Verma module of $\mathrm{F}\oplus\mathrm{NSR}$, $\Delta^{\mathrm{NS}} = \frac{1}{8}(b+b^{-1})^2 - \frac{1}{2}P^2$.

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Proposition (BBFLT)

For generic Δ^{NS} the space $\pi^{\Delta^{NS}}_{F\oplus NSR}$ is isomorphic to $Vir \oplus Vir$ module:

$$\pi^{\Delta^{NS}}_{F\oplus NSR} \cong \bigoplus_{2n\in\mathbb{Z}} \pi^{n}_{Vir\oplus Vir}$$

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- In other words, there exist vectors $|P,n\rangle$:

$$L_0^{(\eta)}|P,n\rangle = \Delta_n^{(\eta)}|P,n\rangle, \quad L_k^{(\eta)}|P,n\rangle = 0, \quad k > 0,\ 2n \in \mathbb{Z},\ \eta = 1,2.$$

We can write that $|P,n\rangle = |\Delta_n^{(1)}\rangle \otimes |\Delta_n^{(2)}\rangle$. The highest weights:

$$\Delta_n^{(1)} = \frac{(b+b^{-1})^2}{8(b^2-1)} + \frac{(P+2nb)^2}{2(b^2-1)}, \quad \Delta_n^{(2)} = \frac{(b+b^{-1})^2}{8(b^{-2}-1)} + \frac{(P+2nb^{-1})^2}{2(b^{-2}-1)},$$

Whittaker vector for Vir

- Whittaker vector for Vir defined by: $|W(q)\rangle = \sum_{N=0}^{\infty} q^{\frac{\Delta+N}{2}}|N\rangle$

  $$L_0|N\rangle = (\Delta + N)|N\rangle, \quad L_1|N\rangle = |N-1\rangle, \quad N > 0, \quad L_k|N\rangle = 0, k > 1$$

  This relation can be rewritten as

  $$L_1|W(q)\rangle = q^{1/2}|W(q)\rangle, \quad L_k|W(q)\rangle = 0, k > 1.$$

  First examples are: $|0\rangle = |\Delta\rangle$, $|1\rangle = \frac{1}{2\Delta}L_{-1}|\Delta\rangle$.

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- The Whittaker limit of the four point conformal block defined by:

  $$\mathcal{F}_c(\Delta; q) = \langle W(q), W(q)\rangle = \sum_{N=0}^{\infty}\langle N, N\rangle q^{\Delta+N}$$

Whittaker vector for F $\oplus$ NSR

Whittaker vector for NSR defined by: $|W(q)_{\mathrm{NSR}}\rangle = \sum_{N=0}^{\infty} q^{\frac{\Delta+N}{2}} |N\rangle^{\mathrm{NS}}$

$$G_{1/2}|W(q)_{\mathrm{NSR}}\rangle = q^{1/4}|W_{\mathrm{NSR}}\rangle, \quad G_s|W(q)_{\mathrm{NSR}}\rangle = 0, s > \frac{1}{2}.$$

Whittaker vector for F $\oplus$ NSR defined as the tensor product

$$|W(q)_{\mathrm{F}\oplus\mathrm{NSR}} = |1\rangle \otimes |W(q)_{\mathrm{NSR}}\rangle,$$

where $|1\rangle$ is a vacuum: $f_r|1\rangle = 0,\ r > 0$.
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$$\mathcal{F}_{c_{\mathrm{NS}}}(\Delta^{\mathrm{NS}}|q) = \langle W_{\mathrm{F}\oplus\mathrm{NSR}}(q)|W_{\mathrm{F}\oplus\mathrm{NSR}}(q)\rangle = \langle W_{\mathrm{NSR}}(q)|W_{\mathrm{NSR}}(q)\rangle.$$

Whittaker vector decomposition

- Decomposition of $\mathsf{F} \oplus \mathsf{NSR}$ representation provides decomposition Whittaker vector:

$$|W_{\mathsf{F}\oplus\mathsf{NSR}}(q)\rangle = \sum_{2n\in\mathbb{Z}} |v(q)\rangle_n,$$

where $|v(q)\rangle_n \in \pi^n_{\mathsf{Vir}\oplus\mathsf{Vir}}$. It turns out that $|v(q)\rangle_n$ is Whittaker vector for algebra $\mathsf{Vir} \oplus \mathsf{Vir}$:

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Proposition

$$|W_{F\oplus NSR}(q)\rangle = \sum_{2n\in\mathbb{Z}} l_n(P,b)\left(|W^{(1)}\rangle_n(\beta^{(1)}q)\otimes|W^{(2)}\rangle_n(\beta^{(2)}q)\right).$$

Here $|W^{(1)}\rangle_n \otimes |W^{(2)}\rangle_n$ — tensor product of Whittaker vectors, $\beta^{(1)} = \left(\frac{b^{-1}}{b^{-1}-b}\right)^2$, $\beta^{(2)} = \left(\frac{b}{b-b^{-1}}\right)^2$.

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- Taking the scalar square of we have:
$$\mathcal{F}_{c^{\mathsf{NS}}}(\Delta^{\mathsf{NS}}|q) = \sum_{2n\in\mathbb{Z}} l_n^2(P,b)\mathcal{F}_n^{(1)}\mathcal{F}_n^{(2)},$$
where $\mathcal{F}_n^{(1)} = \mathcal{F}_{c^{(1)}}(\Delta_n^{(1)}|\beta^{(1)}q)$, $\mathcal{F}_n^{(2)} = \mathcal{F}_{c^{(2)}}(\Delta_n^{(2)}|\beta^{(2)}q)$.

Operator H

Let us introduce operator H:

$$H = bL_0^{(1)} + b^{-1} L_0^{(2)},$$

and define $\widehat{\mathcal{F}_k}$ as:

$$\widehat{\mathcal{F}_{\text{NS}}} = \langle W_{\text{F}\oplus\text{NSR}} | e^{\alpha H} | W_{\text{F}\oplus\text{NSR}} \rangle = \sum_{k=0}^{+\infty} \langle W_{\text{F}\oplus\text{NSR}} | H^k | W_{\text{F}\oplus\text{NSR}} \rangle \frac{\alpha^k}{k!} = \sum_{k=0}^{+\infty} \widehat{\mathcal{F}_k} \frac{\alpha^k}{k!}$$

Clearly $\widehat{\mathcal{F}_0} = \mathcal{F}_{c^{\text{NS}}}(\Delta^{\text{NS}}|q)$.

Calculation of $\widehat{\mathcal{F}_k}$ (I)

- We have $H = bL_0^{(1)} + b^{-1}L_0^{(2)}$,
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- $$\langle W_{\mathrm{F}\oplus\mathrm{NSR}}(q)|e^{\alpha H}|W_{\mathrm{F}\oplus\mathrm{NSR}}(q)\rangle = \sum_{2n\in\mathbb{Z}} l_n^2(P,b)\times$$
$$\times\left\langle W_n^{(1)}(\beta^{(1)}q)\right| e^{\alpha b L_0^{(1)}} \left|W_n^{(1)}(\beta^{(1)}q)\right\rangle \left\langle W_n^{(2)}(\beta^{(2)}q)\right| e^{\alpha b^{-1} L_0^{(2)}} \left|W_n^{(2)}(\beta^{(2)}q)\right\rangle.$$

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- $\widehat{\mathcal{F}_k} = \sum_{2n\in\mathbb{Z}} l_n^2(P,b)\cdot D^k_{b,b^{-1}[\log q]}(\mathcal{F}_n^{(1)},\mathcal{F}_n^{(2)})$,
where generalized Hirota: $f(e^{\epsilon_1\alpha}q)g(e^{\epsilon_2\alpha}q) = \sum_{n=0}^{+\infty} D^n_{\epsilon_1,\epsilon_2[\log q]}(f(q),g(q))\frac{\alpha^n}{n!}$

Calculation of $\widehat{\mathcal{F}_k}$ (II)

- On the other hand we can rewrite operator H in terms of F $\oplus$ NSR generators:

$$H = (b + b^{-1}) \sum_{r \in \mathbb{Z}-1/2} r : f_{-r} f_r : - \sum_{r \in \mathbb{Z}-1/2} f_{-r} G_r,$$

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- Taking scalar product we get

$$\widehat{\mathcal{F}_0} = \mathcal{F}_{\mathsf{NS}}, \quad \widehat{\mathcal{F}_2} = -q^{1/2}\mathcal{F}_{\mathsf{NS}}, \quad \widehat{\mathcal{F}_4} = q^{1/2}(2q\frac{d}{dq}\mathcal{F}_{\mathsf{NS}} - q^{1/2}\mathcal{F}_{\mathsf{NS}}) - (b+b^{-1})^2 q^{1/2}\mathcal{F}_{\mathsf{NS}},$$

where $\mathcal{F}_{\mathsf{NS}}$ denotes $\mathcal{F}_{c^{\mathsf{NS}}}(\Delta^{\mathsf{NS}}|q)$.

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- Therefore we get an equation $\widehat{\mathcal{F}_4} + 2q\frac{d}{dq}\widehat{\mathcal{F}_2} - (1 + (b + b^{-1})^2)\widehat{\mathcal{F}_2} + q\widehat{\mathcal{F}_0} = 0$.

Conformal blocks relations

- Introduce operator.

$$D_b^{III} = D^4_{b,b^{-1}[\log q]} + 2q\frac{d}{dq}D^2_{b,b^{-1}[\log q]} - (1+(b+b^{-1})^2)D^2_{b,b^{-1}[\log q]} + qD^0_{b,b^{-1}[\log q]}$$

We proved that:

$$\sum_{2n\in\mathbb{Z}} l_n^2(P,b)\cdot D_b^{III}\left(\mathcal{F}_{c^{(1)}}(\Delta_n^{(1)}|\beta^{(1)}q), \mathcal{F}_{c^{(2)}}(\Delta_n^{(2)}|\beta^{(2)}q)\right) = 0$$

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- We want to specialize central charges to 1. We set $b = i$. Other parameters are specified by:

$$q = 4t; \quad P = 2i\sigma,\ P^{(\eta)} = i\sigma,\ \eta = 1,2, \quad \Delta_n^{(1)} = (\sigma+n)^2,\ \Delta_n^{(2)} = (\sigma-n)^2.$$

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- We proved:

$$\sum_{2n\in\mathbb{Z}} l_n(2i\sigma, i)^2 \times D^{III}\Big(\mathcal{F}((\sigma+n)^2|t), \mathcal{F}((\sigma-n)^2|t)\Big) = 0,$$

Structure constants

Proposition

$$l_n(P,b) = \frac{(-1)^n 2^{2n^2} (\beta^{(1)})^{-\Delta_n^{(1)}/2} (\beta^{(2)})^{-\Delta_n^{(2)}/2}}{\sqrt{s_{even}(2P,2n) s_{even}(2P+b+b^{-1},2n)}},$$

where

$$s_{even}(x,n) = \prod_{\substack{i,j\geq 0,\ i+j<2n \\ i+j\equiv 0 \bmod 2}} (x + ib + jb^{-1}),$$

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- The calculation $l_n(P,b)$ is the last step inthe proof of AGT for NSR algebra. (mostly based on the proof for Virasoro: Alba, Fateev, Litvinov, Tarnopolsky, and previous works on NSR V. Belavin, B. Feigin; G. Bonelli, K. Maruyoshi, A. Tanzini; [BBFLT].)

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- Using the recurrence equation on G(q) one can prove for $2n \in \mathbb{Z}$:

$$\frac{C(\sigma+n)C(\sigma-n)}{C(\sigma)^2} = (\prod_{k=1}^{2|n|-1} (k^2-4\sigma^2)^{2(2|n|-k)}(4\sigma^2)^{2|n|})^{-1} = 4^{-\Delta^{NS}}(-1)^{2n} l_n(2i\sigma,i)^2$$

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- Questions
  - Possible generalizations: many points, higher genus, higher ranks.
  - Geometrical meaning
  - Deformation of Painlevé equations.
  - Litvinov, Lukyanov, Nekrasov, Zamolodchikov connection: $c = \infty$ and Painlevé.